

Introduction to Logic

Lecture Notes, Unit 4

General caveat about online notes: These notes are provided in case you miss anything. Reading them on their own is poor substitute for attending class, for a number of reasons:

(i) What is taught in this course are *processes* for solving certain problems. These notes do not show the processes, but only the end results, and cannot substitute for seeing the processes in action, and may be difficult to understand on their own.

(ii) These notes constitute only what I'm likely to write or project on the screen; they do not contain nearly everything I say in lecture that it might be helpful for you to know.

(iii) These notes are from last time I taught the course, and I reserve the right to change things this semester.

PREDICATE LOGIC DERIVATIONS

In this unit, we use the method of *derivations*, like in unit 2, as a means to demonstrate that arguments in predicate logic are valid.

This method is particularly important for predicate logic, because, unlike sentential logic, there is no alternative method such as truth tables to use to show that an argument is valid.

The basic idea of a derivation in predicate logic is the same as for sentential logic.

Moreover, all the rules of derivation you learned for sentential logic (&O, &I, vO, vI, ↔O, ↔I, →O, DN, etc.) carry over to predicate logic.

The only difference is that, although in stating the rules, we use single letters to represent places where we can "plug in" formulas, the formulas themselves will never consist of single letters.

At minimum, they will consist of a capital letter and lowercase letter. They may also contain quantifiers and/or other logical operators.

Example:

1.	$Fa \ \& \ \exists xGx$	Pr
2.	$\exists xGx \leftrightarrow Hb$	Pr
3.	SHOW: $Fa \ \& \ Hb$	DD
4.	Fa	1&O
5.	$\exists xGx$	1&O
6.	$\exists xGx \rightarrow Hb$	2 ↔O
7.	Hb	5,6 →O
8.	$Fa \ \& \ Hb$	4,7 &I

The techniques of ID and CD also carry over.

1.	$(Ma \vee Na) \rightarrow \forall xOx$	Pr
2.	SHOW: $Na \rightarrow \forall xOx$	CD
3.	Na	Ass
4.	SHOW: $\forall xOx$	DD
5.	$Ma \vee Na$	3 vI
6.	$\forall xOx$	1,5 →O

Warning: we can only use CD for SHOW lines whose "main operator" is the if-then (→). This does not include statements that have if-thens within the scope of quantifiers.

If our show line is:

?. SHOW: $\forall x(Nx \rightarrow Ox)$

We cannot use CD, or, at least, not right away. We could, however use it for:

?. SHOW: $\forall xNx \rightarrow \forall xOx$

Universal-Out

We need new rules to use for our new logical signs: the quantifiers.

Again, these can be divided into 'in' rules and 'out' rules. Our first such rule is called "Universal Out".

UNIVERSAL OUT (\forall O)

$\forall x...x...$ (or any other variable besides 'x')

...a... (or any other name)

(The quantifier must extend over the whole statement. We cannot apply this rule, or any other rule, to part of a line.)

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We start with something that says that some formula is true *for all things* x . (Or, for all things y ...) Therefore, that formula would be true for any particular thing. So we can replace the variable that goes with the quantifier everywhere it occurs afterwards with the same name, and the result will be true. (We must replace every occurrence of the variable letter with the same name.)

For example, consider this argument:
For everything, if it is human, then it is mortal.
Therefore, if Socrates is human, then Socrates is mortal.

This argument essentially has this form:

$\forall x(Hx \rightarrow Mx)$
 $Hs \rightarrow Ms$

This is a valid reasoning step by $\forall O$.

Some example problems:

- | | | |
|----|--------------------------------|---------------------|
| 1. | $\forall x(Fx \rightarrow Gx)$ | Pr |
| 2. | $\forall x(Gx \rightarrow Hx)$ | Pr |
| 3. | Fd | Pr |
| 4. | SHOW: Hd | DD |
| 5. | Fd \rightarrow Gd | 1 $\forall O$ |
| 6. | Gd \rightarrow Hd | 2 $\forall O$ |
| 7. | Gd | 3,5 $\rightarrow O$ |
| 8. | Hd | 6,7 $\rightarrow O$ |

[We must do the $\forall O$ steps before $\rightarrow O$ steps.]

We can do $\forall O$ to any letter we like, and even to multiple letters within the same derivation. The only thing we are restricted from doing is replacing different occurrences of the variable with different letters within *one* step.

- | | | |
|-----|--------------------------------------|---------------------|
| 1. | $\forall x(Fx \& Gx)$ | Pr |
| 2. | $(Fa \& Gb) \rightarrow \forall yHy$ | Pr |
| 3. | SHOW: Hc | DD |
| 4. | Fa & Ga | 1 $\forall O$ |
| 5. | Fb & Gb | 1 $\forall O$ |
| 6. | Fa | 4 & O |
| 7. | Gb | 5 & O |
| 8. | Fa & Gb | 6,7 & I |
| 9. | $\forall yHy$ | 2,8 $\rightarrow O$ |
| 10. | Hc | 9 $\forall O$ |

Here's a use of $\forall O$ within an ID.

- | | | |
|----|----------------------------|----------------|
| 1. | $\sim Fa \vee \exists xGx$ | Pr |
| 2. | $\sim \exists xGx$ | Pr |
| 3. | SHOW: $\sim \forall xFx$ | ID |
| 4. | $\forall xFx$ | Ass |
| 5. | SHOW: \times | DD |
| 6. | $\sim Fa$ | 1,2 $\vee O$ |
| 7. | Fa | 4 $\forall O$ |
| 8. | \times | 6,7 $\times I$ |

Take multiple quantifiers in the same statement one at a time.

- | | | |
|-----|---------------------------------|---------------------|
| 1. | $\forall x\forall yRxy$ | Pr |
| 2. | $\forall x(Rxx \rightarrow Fx)$ | Pr |
| 3. | SHOW: Fa & Rba | DD |
| 4. | $\forall yRby$ | 1 $\forall O$ |
| 5. | $\forall yRay$ | 1 $\forall O$ |
| 6. | Rba | 4 $\forall O$ |
| 7. | Raa | 5 $\forall O$ |
| 8. | Raa \rightarrow Fa | 2 $\forall O$ |
| 9. | Fa | 7,8 $\rightarrow O$ |
| 10. | Fa & Rba | 6,9 & O |

Existential In

Our next rule deals with existential quantifiers. Here, we move in the opposite direction. We begin with a statement about some particular individual: a, b, c, etc. We then conclude that the statement is true about *something* (or someone).

EXISTENTIAL IN ($\exists I$)

...a... (or any other letter)

$\exists x$...x... (or any other variable not already in the statement)

Here, you replace one or more occurrences of the letter with a variable, and introduce an existential quantifier, which will extend over the whole statement. (You may need to add parentheses.) If the letter x already occurs in the statement, you will want to use y or z , etc., as required.

Consider the argument:
The Bandersnatch is frumious.
Therefore, something is frumious.

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This has the form:

Fb
 $\exists xFx$

This is a valid step of $\exists I$ reasoning.

Some examples:

- | | | |
|----|-------------------------|---------------|
| 1. | $\forall x(Fx \vee Gx)$ | Pr |
| 2. | $\sim Fb$ | Pr |
| 3. | SHOW: $\exists xGx$ | DD |
| 4. | $Fb \vee Gb$ | 1 $\forall O$ |
| 5. | Gb | 2,4 $\vee O$ |
| 6. | $\exists xGx$ | 5 $\exists I$ |

- | | | |
|----|------------------------------------|-----------------------|
| 1. | $\forall x(Fx \leftrightarrow Gx)$ | Pr |
| 2. | Fb | Pr |
| 3. | SHOW: $\exists x(Gx \vee Hx)$ | DD |
| 4. | $Fb \leftrightarrow Gb$ | 1 $\forall O$ |
| 5. | $Fb \rightarrow Gb$ | 4 $\leftrightarrow O$ |
| 6. | Gb | 2,5 $\rightarrow O$ |
| 7. | $Gb \vee Hb$ | 6 $\vee I$ |
| 8. | $\exists x(Gx \vee Hx)$ | 7 $\exists I$ |

Unlike $\forall O$, we do not need to replace all occurrences of the letter; we can replace only one—or more—as required.

- | | | |
|----|------------------------------------|---------------------|
| 1. | $\forall x(Txx \rightarrow Sx)$ | Pr |
| 2. | $\sim Se$ | Pr |
| 3. | SHOW: $\exists x\exists y\sim Txy$ | DD |
| 4. | $Tee \rightarrow Se$ | 1 $\forall O$ |
| 5. | $\sim Tee$ | 2,4 $\rightarrow O$ |
| 6. | $\exists y\sim Tey$ | 5 $\exists I$ |
| 7. | $\exists x\exists y\sim Txy$ | 6 $\exists I$ |

- | | | |
|-----|--|---------------------|
| 1. | $\forall x(Fx \rightarrow Gx)$ | Pr |
| 2. | $\forall x(Fx \rightarrow \sim Hxx)$ | Pr |
| 3. | SHOW: $Fa \rightarrow \exists x(Gx \ \& \ \sim Hxa)$ | CD |
| 4. | Fa | Ass |
| 5. | SHOW: $\exists x(Gx \ \& \ \sim Hxa)$ | DD |
| 6. | $Fa \rightarrow Ga$ | 1 $\forall O$ |
| 7. | Ga | 4,6 $\rightarrow O$ |
| 8. | $Fa \rightarrow \sim Haa$ | 2 $\forall O$ |
| 9. | $\sim Haa$ | 4,8 $\rightarrow O$ |
| 10. | $Ga \ \& \ \sim Haa$ | 7,8 $\& I$ |
| 11. | $\exists x(Gx \ \& \ \sim Hxa)$ | 10 $\exists I$ |

Existential Out

The next rule we are going to learn is trickier. Last time, we learned $\forall O$, which allows us to drop off a universal quantifier and replace the variable with any letter we like. Of course, a universal quantifier is used to say that something is true about *everything* there is. An existential quantifier just says that something is true for *at least one* thing, but it doesn't tell us what that thing is.

Therefore, we can't drop off an existential quantifier to any letter we like. Instead, we simply make up a *new name* for the person or thing that makes the existentially quantified statement true. Then we use that name. The process is somewhat like this.

Something lives on Venus.
(Let's arbitrarily call that something ALF.)
Therefore, ALF lives on Venus.

This argument has this form.

$\exists xLxv$
 Lav

Our rule of existential out looks like this:

EXISTENTIAL OUT ($\exists O$)
 $\exists x...x...$ (or other variable)
 $...a...$ (where 'a' is a NEW NAME, not used earlier in the derivation.)

We must pick a new letter. We cannot assume that the thing making the quantified statement true is the same thing as anything we already know something about. So if we already have 'a' earlier in our derivation, we must use 'b'. If we already have 'a' and 'b', we must use 'c', and so forth.

An example derivation.

- | | | |
|----|--------------------------------|---------------------|
| 1. | $\forall x(Fx \rightarrow Gx)$ | Pr |
| 2. | $\exists xFx$ | Pr |
| 3. | SHOW: $\exists xGx$ | DD |
| 4. | Fa | 2 $\exists O$ |
| 5. | $Fa \rightarrow Ga$ | 1 $\forall O$ |
| 6. | Ga | 4,5 $\rightarrow O$ |
| 7. | $\exists xGx$ | 6 $\exists I$ |

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We can use 'a' at line 4 since it doesn't appear above.

This problem shows that it's almost always better to do $\exists O$ steps before $\forall O$ steps.

Notice that if we had done the $\forall O$ step first, we would have had to use a different letter—say "b"—for the $\exists O$ step, and then we wouldn't be able to do $\rightarrow O$!

Some more examples:

1.	$\exists xFx \rightarrow \forall x(\sim Fx \rightarrow \sim Gx)$	Pr
2.	$\exists x(\sim Fx \ \& \ \sim Rax)$	Pr
3.	SHOW: $Fb \rightarrow \exists x\sim Gx$	CD
4.	Fb	Ass
5.	SHOW: $\exists x\sim Gx$	DD
6.	$\sim Fc \ \& \ \sim Rac$	2 $\exists O$
7.	$\sim Fc$	6 & O
8.	$\exists xFx$	4 $\exists I$
9.	$\forall x(\sim Fx \rightarrow \sim Gx)$	1,8 $\rightarrow O$
10.	$\sim Fc \rightarrow \sim Gc$	9 $\forall O$
11.	$\sim Gc$	7,10 $\rightarrow O$
12.	$\exists x\sim Gx$	11 $\exists I$

[We use 'c' since 'a' and 'b' already appear.]

1.	$\forall x(Fx \leftrightarrow Gx)$	Pr
2.	$\forall x\sim Fx$	Pr
3.	SHOW: $\sim \exists xGx$	ID
4.	$\exists xGx$	Ass
5.	SHOW: \times	DD
6.	Ga	4 $\exists O$
7.	$Fa \leftrightarrow Ga$	1 $\forall O$
8.	$\sim Fa$	2 $\forall O$
9.	$Ga \rightarrow Fa$	7 $\leftrightarrow O$
10.	Fa	6,9 $\rightarrow O$
11.	\times	8,10 $\times I$

Universal Derivation

There is no rule of "Universal In". Instead, we have a new form of derivation. (Analogously, remember we did not have a rule of $\rightarrow I$, only CD.)

It takes a lot to *prove* that a certain formula holds of *everything* in the universe. Certainly,

it is not enough to prove that the formula holds of some particular things, or some particular things you know things about.

However, suppose you made up a new name, 'Frabjous', and you could prove that some formula was true of 'Frabjous', without knowing *anything else at all* about 'Frabjous'. It must be that you could give the same proof about anything else in the world, since it couldn't be anything special about 'Frabjous'.

This is the basic idea behind universal derivation. If you want to SHOW: $\forall x...x...$, it suffices to show $...a...$, where 'a' is a *new*, made-up name that doesn't occur anywhere above in your derivation.

Unlike CD and ID, **universal derivation** (UD), does not involve making an assumption.

To set one up, we simply write in a new SHOW line, making use of a *new letter*.

Here's an example.

1.	$\forall x(Fx \rightarrow Gx)$	Pr
2.	$\forall xFx$	Pr
3.	SHOW: $\forall x(Fx \ \& \ Gx)$	

First we write "UD" on the right. Then, without any assumption line, our next line will be a *new* show line where we drop off the quantifier, and replace all the occurrences of the variable with the same new name:

1.	$\forall x(Fx \rightarrow Gx)$	Pr
2.	$\forall xFx$	Pr
3.	SHOW: $\forall x(Fx \ \& \ Gx)$	UD
4.	SHOW: $Fa \ \& \ Ga$	

Once again, the idea is that, if I can show the formula about some arbitrary entity I know nothing about, I can prove the universally quantified formula.

Notice that 4 is a show line. I *can't use it!* I still don't really know anything about this entity "a". I'm just giving myself something

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new to prove. I can prove it using the other techniques: CD, ID, or DD.

1.	$\forall x(Fx \rightarrow Gx)$	Pr
2.	$\forall xFx$	Pr
3.	SHOW: $\forall x(Fx \& Gx)$	UD
4.	SHOW: $Fa \& Ga$	DD
5.	$Fa \rightarrow Ga$	1 $\forall O$
6.	Fa	2 $\forall O$
7.	Ga	5,6 $\rightarrow O$
8.	$Fa \& Ga$	5,8 $\& I$

Now that I've *shown* $Fa \& Ga$, and crossed off the 'SHOW: ' for that line, I remember that I used the letter 'a' arbitrarily—I could have done the same derivation for any other individual. Therefore, I am entitled to cross off the 'SHOW: ' at line 3, by UD.

1.	$\forall x(Fx \rightarrow Gx)$	Pr
2.	$\forall xFx$	Pr
3.	SHOW: $\forall x(Fx \& Gx)$	UD
4.	SHOW: $Fa \& Ga$	DD
5.	$Fa \rightarrow Ga$	1 $\forall O$
6.	Fa	2 $\forall O$
7.	Ga	5,6 $\rightarrow O$
8.	$Fa \& Ga$	5,8 $\& I$

1.	$\forall x(Gx \rightarrow \sim \exists yFy)$	Pr
2.	$Fa \& Hb$	Pr
3.	SHOW: $\forall x \sim Gx$	UD
4.	SHOW: $\sim Gc$	DD
5.	Fa	2 $\& O$
6.	$\exists yFy$	5 $\exists I$
7.	$Gc \rightarrow \sim \exists yFy$	1 $\forall O$
8.	$\sim \sim \exists yFy$	6 DN
9.	$\sim Gc$	7,8 $\rightarrow O$

1.	$\forall x(Fx \rightarrow \sim Hx)$	Pr
2.	$\forall x(Gx \vee Hx)$	Pr
3.	SHOW: $\forall x(Fx \rightarrow Gx)$	UD
4.	SHOW: $Fa \rightarrow Ga$	CD
5.	Fa	Ass
6.	SHOW: Ga	DD
7.	$Fa \rightarrow \sim Ha$	1 $\forall O$
8.	$Ga \vee Ha$	2 $\forall O$
9.	$\sim Ha$	5,7 $\rightarrow O$
10.	Ga	8,9 $\vee O$

Here's a UD within a CD:

1.	$\forall x(Fx \rightarrow Rxa)$	Pr
2.	$\forall x(\exists yRya \rightarrow Hx)$	Pr
3.	SHOW: $\exists x(Fx \& Gx) \rightarrow \forall xHx$	CD
4.	$\exists x(Fx \& Gx)$	Ass
5.	SHOW: $\forall xHx$	UD
6.	SHOW: Hb	DD
7.	$Fc \& Gc$	4 $\exists O$
8.	$Fc \rightarrow Rca$	1 $\forall O$
9.	$\exists yRya \rightarrow Hb$	2 $\forall O$
10.	Fc	7 $\& O$
11.	Rca	8,10 $\rightarrow O$
12.	$\exists yRya$	11 $\exists I$
13.	Hb	9,12 $\rightarrow O$

Here's an ID within a UD.

1.	$\forall x(Fx \leftrightarrow \sim Gx)$	Pr
2.	$\sim \exists xGx$	Pr
3.	SHOW: $\forall xFx$	UD
4.	SHOW: Fa	ID
5.	$\sim Fa$	Ass
6.	SHOW: \times	DD
7.	$Fa \leftrightarrow \sim Ga$	1 $\forall O$
8.	$\sim Ga \rightarrow Fa$	7 $\leftrightarrow O$
9.	$\sim \sim Ga$	5,8 $\rightarrow O$
10.	Ga	9 DN
11.	$\exists xGx$	10 $\exists I$
12.	\times	2,11 $\times I$

Here's an ID within a UD within a CD.

1.	$\forall x(Fx \rightarrow \forall yHy)$	Pr
2.	$\sim \exists x(Hx \& Gx)$	Pr
3.	SHOW: $\exists xFx \rightarrow \forall x \sim Gx$	CD
4.	$\exists xFx$	Ass
5.	SHOW: $\forall x \sim Gx$	UD
6.	SHOW: $\sim Ga$	ID
7.	Ga	Ass
8.	SHOW: \times	DD
9.	Fb	4 $\exists O$
10.	$Fb \rightarrow \forall yHy$	1 $\forall O$
11.	$\forall yHy$	9,10 $\rightarrow O$
12.	Ha	11 $\forall O$
13.	$Ga \& Ha$	7,12 $\& I$
14.	$\exists x(Hx \& Gx)$	13 $\exists I$
15.	\times	2,14 $\times I$

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Negation Rules:

If a quantifier has a negation in front of it, we cannot use $\exists O$ or $\forall O$. Remember our rules apply only to whole lines.

However, we now introduce new rules that allow us to make use of statements that begin with negations of quantifiers in front. (These rules are redundant, but very helpful.)

The rules are these.

NEGATION-UNIVERSAL OUT ($\sim\forall O$)

$\sim\forall x...x...$ (or other variable)

$\exists x\sim...x...$

NEGATION-EXISTENTIAL OUT ($\sim\exists O$)

$\sim\exists x...x...$ (or other variable)

$\forall x\sim...x...$

A little reflection shows these rules to be valid.

If it's not true that everything is F, then something must not be F.

If it's not true that something is F, then everything must not be F.

Consider these arguments:

Not everything is colored.

Therefore, something is not colored.

It's not true that something is omnipotent.

Therefore, everything is not omnipotent.

These arguments exemplify the patterns of reasoning above.

Another way of putting these rules is that you can "push" a negation through a quantifier if you *change* the quantifier.

However, you *must* change the quantifier. One cannot go from $\sim\forall x...x...$ to $\forall x\sim...x...$!

Some examples:

1.	$\sim\exists x(Fx \& Gx)$	Pr
2.	$\exists xFx$	Pr
3.	SHOW: $\exists x\sim Gx$	DD
4.	$\forall x\sim(Fx \& Gx)$	1 $\sim\exists O$
5.	Fa 2	$\exists O$
6.	$\sim(Fa \& Ga)$	4 $\forall O$
7.	Fa $\rightarrow \sim Ga$	7 $\sim\&O$
8.	$\sim Ga$	5,7 $\rightarrow O$
9.	$\exists x\sim Gx$	8 $\exists I$

1.	$\forall x(Fx \rightarrow Gx)$	Pr
2.	$\sim\forall x(Hx \rightarrow Gx)$	Pr
3.	SHOW: $\sim\forall x(Hx \rightarrow Fx)$	ID
4.	$\forall x(Hx \rightarrow Fx)$	Ass
5.	SHOW: *	DD
6.	$\exists x\sim(Hx \rightarrow Gx)$	2 $\sim\forall O$
7.	$\sim(Ha \rightarrow Ga)$	6 $\exists O$
8.	Ha $\& \sim Ga$	7 $\sim\rightarrow O$
9.	Ha	8 $\&O$
10.	$\sim Ga$	8 $\&O$
11.	Fa $\rightarrow Ga$	1 $\forall O$
12.	Ha $\rightarrow Fa$	4 $\forall O$
13.	Fa	9,12 $\rightarrow O$
14.	Ga	11,13 $\rightarrow O$
15.	*	10,14 $\times I$

1.	$\exists xGx \rightarrow \forall x(Hx \& Ix)$	Pr
2.	SHOW: $\forall x\sim Gx \vee \forall x(Fx \vee Ix)$	ID
3.	$\sim[\forall x\sim Gx \vee \forall x(Fx \vee Ix)]$	Ass
4.	SHOW: *	DD
5.	$\sim\forall x\sim Gx$	3 $\sim\forall O$
6.	$\sim\forall x(Fx \vee Ix)$	3 $\sim\forall O$
7.	$\exists x\sim\sim Gx$	5 $\sim\exists O$
8.	$\exists x\sim(Fx \vee Ix)$	6 $\sim\exists O$
9.	$\sim\sim Ga$	7 $\exists O$
10.	$\sim(Fb \vee Ib)$	8 $\exists O$
11.	Ga	9 DN
12.	$\exists xGx$	11 $\exists I$
13.	$\forall x(Hx \& Ix)$	1,12 $\rightarrow O$
14.	Hb $\& Ib$	13 $\forall O$
15.	Ib	14 $\&O$
16.	$\sim Ib$	10 $\sim\forall O$
17.	*	15,16 $\times I$

Yes, we get a double negation at line 7.

The rule only pushes the negation through: it cannot eliminate it.

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1.	$\sim\forall xFx \vee \sim\exists xGx$	Pr
2.	$\exists y\sim Fy \rightarrow \sim\exists x(Hx \& Ix)$	Pr
3.	SHOW: $Ga \rightarrow \forall x(Hx \rightarrow \sim Ix)$	CD
4.	Ga	Ass
5.	SHOW: $\forall x(Hx \rightarrow \sim Ix)$	UD
6.	SHOW: $Hb \rightarrow \sim Ib$	CD
7.	Hb	Ass
8.	SHOW: $\sim Ib$	DD
9.	$\exists xGx$	4 EI
10.	$\sim\sim\exists xGx$	9 DN
11.	$\sim\forall xFx$	1,10 vO
12.	$\exists x\sim Fx$	11 $\sim\forall$ O
13.	$\sim Fc$	12 \exists O
14.	$\exists y\sim Fy$	13 EI
15.	$\sim\exists x(Hx \& Ix)$	2,14 \rightarrow O
16.	$\forall x\sim(Hx \& Ix)$	15 $\sim\exists$ O
17.	$\sim(Hb \& Ib)$	16 \forall O
18.	$Hb \rightarrow \sim Ib$	17 $\sim\rightarrow$ O
19.	$\sim Ib$	7,18 \rightarrow O

Multiple Quantifiers, Relational Quantification

There is nothing special you need to know to do problems that contain multiple quantifiers. Simply take the quantifiers one at a time, and treat the letters that occur in different places as different when they're different, and the same when they're the same.

Also remember you can't apply the rules to parts of lines.

Here's a problem using two universal derivations, one right after another.

1.	$\sim\exists x\exists y\sim Rxy$	Pr
2.	SHOW: $\forall x\forall yRxy$	UD
3.	SHOW: $\forall yRay$	UD
4.	SHOW: Rab	DD
5.	$\forall x\sim\exists y\sim Rxy$	1 $\sim\exists$ O
6.	$\sim\exists y\sim Ryb$	5 \forall O
7.	$\forall y\sim\sim Ryb$	6 \exists O
8.	$\sim\sim Rab$	7 \forall O
9.	Rab	8 DN

At line 3, you only change the xs to a, and at line 4 you only change the ys to b. We push negations through one at a time as well.

More examples:

1.	$\exists x\forall y(Fx \vee Gy)$	Pr
2.	SHOW: $\forall x(\sim Gx \rightarrow \exists yFy)$	UD
3.	SHOW: $\sim Ga \rightarrow \exists yFy$	CD
4.	$\sim Ga$	Ass
5.	SHOW: $\exists yFy$	DD
6.	$\forall y(Fb \vee Gy)$	1 \exists O
7.	$Fb \vee Ga$	6 \forall O
8.	Fb 4,	7 vO
9.	$\exists yFy$	8 EI
1.	$\forall x[Fx \rightarrow \sim\exists y(Gy \& Ryx)]$	Pr
2.	Ga	Pr
3.	SHOW: $Fa \rightarrow \sim Raa$	CD
4.	Fa	Ass
5.	SHOW: $\sim Raa$	DD
6.	$Fa \rightarrow \sim\exists y(Gy \& Rya)$	1 \forall O
7.	$\sim\exists y(Gy \& Rya)$	4,6 \rightarrow O
8.	$\forall y\sim(Gy \& Rya)$	7 $\sim\exists$ O
9.	$\sim(Ga \& Raa)$	8 $\sim\forall$ O
10.	$Ga \rightarrow \sim Raa$	9 $\sim\&$ O
11.	$\sim Raa$	2,10 \rightarrow O

Only the xs change at line 6.

Here's my favorite problem in the world.

*Everyone loves a lover.
Someone loves someone.
Therefore, everyone loves everyone.*

1.	$\forall x(\exists zLxz \rightarrow \forall yLyx)$	Pr
2.	$\exists x\exists yLxy$	Pr
3.	SHOW: $\forall x\forall yLxy$	UD
4.	SHOW: $\forall yLay$	UD
5.	SHOW: Lab	DD
6.	$\exists yLcy$	2 \exists O
7.	Lcd	6 \exists O
8.	$\exists zLcz \rightarrow \forall yLyc$	1 \forall O
9.	$\exists zLcz$	7 EI
10.	$\forall yLyc$	8,9 \rightarrow O
11.	Lbc	10 \forall O
12.	$\exists zLbz \rightarrow \forall yLyb$	1 \forall O
13.	$\exists zLbz$	11 EI
14.	$\forall yLyb$	12,13 \rightarrow O
15.	Lab	14 \forall O

The key to the problem are realizing that you can use the first premise more than once,

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and that you can juggle the names around as needed.

We can show everyone loves everyone by showing that, for arbitrary “people” a and b , a loves b . We do this by going through the universal statement twice. Since someone loves someone, we give them names, and call them c and d . Since c loves d , c is a lover. Since c is a lover, everyone loves c . If everyone loves c , everyone, including b , is a lover. If b is a lover, everyone, including a , loves b . So a loves b , and everyone loves everyone.

Strategies and Steps

This chart suggests a kind of derivation to do depending on your SHOW line.

SHOW: line	Form	Try	Assume	SHOW:
simple statement	Fa	ID	$\sim Fa$	✖
negation (any kind)	$\sim Fa$	ID	Fa	✖
	$\sim(_ \vee \dots)$		$_ \vee \dots$	
	$\sim(_ \& \dots)$		$_ \& \dots$	
	$\sim(_ \rightarrow \dots)$		$_ \rightarrow \dots$	
	$\sim(_ \leftrightarrow \dots)$		$_ \leftrightarrow \dots$	
	$\sim\exists x \dots x \dots$		$\exists x \dots x \dots$	
	$\sim\forall x \dots x \dots$		$\forall x \dots x \dots$	
disjunction	$_ \vee \dots$	ID	$\sim(_ \vee \dots)$	✖
existential*	$\exists x \dots x \dots$	ID	$\sim\exists x \dots x \dots$	✖
conditional	$_ \rightarrow \dots$	CD	$_$...
universal	$\forall x \dots x \dots$	UD	[none]	...a... [new]

For ‘and’ and ‘iff’ statements, break the problem into two parts, and put them together at the end with $\&I$ or $\leftrightarrow I$.

SHOW line	Form	Try	Assume	Show	
conjunction	$_ \& \dots$	DD			
	SHOW: $_$		ID	$\sim _$	✖
	SHOW: \dots		ID	$\sim \dots$	✖
biconditional	$_ \leftrightarrow \dots$	DD			
	SHOW: $_ \rightarrow \dots$		CD	$_$...
	SHOW: $\dots \rightarrow _$		CD	\dots	$_$

* = most existentials can be shown by DD; however, using ID is often easier, so this is what I recommend.

This is the same as a chart I gave you for unit 2, adding that universals should be proven by UD, and existentials by ID.

There is also a step-by-step procedure I recommend for completing the problems after they’ve been set up.

You do not need to memorize this if you’re not having trouble with the derivations, but for many students, it is a helpful guide.

STEP 1. Look at SHOW: line; consult the table above and set-up the derivation.

STEP 2. Look at the new SHOW: line; if it is SHOW: ✖, proceed to step 3. If it is anything else, repeat step 1 for the new SHOW: line.

STEP 3. Do all $\sim\exists O$ and $\sim\forall O$ steps. (i.e., push any negations through the quantifiers.)

STEP 4. Do all the $\exists O$ steps you can. Use a new letter every time.

STEP 5. For every universal statement, do a $\forall O$ step to every name you already have in the derivation, however many there are.

STEP 6. Apply the DD rules. ($\exists I$, DN , $\forall O$, $\rightarrow O$, $\&O$, $\leftrightarrow O$, etc.)

STEP 7. Look at results from step 6. If you got a contradiction, you can finish the derivation with $\&I$. If you don’t have one, you may need to repeat steps 3-6 until you get one. If you still can’t get one, you may need to try a “desperate measures” strategy. (Remember those?)

Here are some problems that make use of these strategies and directions.

- | | | |
|----|-------------------------------------|-------------------|
| 1. | $\exists x(Hx \rightarrow Fx)$ | Pr |
| 2. | $\forall x(\sim Hx \rightarrow Gx)$ | Pr |
| 3. | SHOW: $\exists x(Fx \vee Gx)$ | ID |
| 4. | $\sim\exists x(Fx \vee Gx)$ | Ass |
| 5. | SHOW: ✖ | DD |
| 6. | $\forall x\sim(Fx \vee Gx)$ | 4 $\sim\exists O$ |
| 7. | $Ha \rightarrow Fa$ | 1 $\exists O$ |
| 8. | $\sim Ha \rightarrow Ga$ | 2 $\forall O$ |

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9.	$\sim(Fa \vee Ga)$	6 $\forall O$
10.	$\sim Fa$	9 $\sim v O$
11.	$\sim Ga$	9 $\sim v O$
12.	$\sim Ha$	7,10 $\rightarrow O$
13.	Ga	8,12 $\rightarrow O$
14.	\ast	11,13 $\ast I$

Lines 1-5 are STEPS 1-2.

Line 6 is STEP 3.

Line 7 is STEP 4.

Lines 8-9 are STEP 5.

Lines 10-13 are STEP 6.

Line 14 is STEP 7.

1.	$\exists x(Rxa \vee Rax)$	Pr
2.	SHOW: $\exists xRxa \vee \exists xRax$	ID
3.	$\sim(\exists xRxa \vee \exists xRax)$	Ass
4.	SHOW: \ast	DD
5.	$Rba \vee Rab$	1 $\exists O$
6.	$\sim \exists xRxa$	3 $\sim v O$
7.	$\sim \exists xRax$	3 $\sim v O$
8.	$\forall x \sim Rxa$	6 $\exists E O$
9.	$\forall x \sim Rax$	6 $\exists E O$
10.	$\sim Raa$	8 $\forall O$
11.	$\sim Rba$	8 $\forall O$
12.	$\sim Rab$	9 $\forall O$
13.	Rab	5,11 $v O$
14.	\ast	12,13 $\ast I$

Lines 1-4 are STEPS 1-2.

STEP 3 is skipped, since there are not negations before quantifiers.

Line 5 is STEP 4.

STEP 5 is skipped.

Lines 6-7 are STEP 6.

STEP 7 tells us to repeat steps 3-6.

Lines 8-9 are STEP 3 repeated.

Lines 10-12 are STEP 5 repeated.

Line 13 is a new STEP 6.

Line 14 is STEP 7.

We did an unnecessary step at line 10, but that's OK.

Biconditionals can make problems quite long, but easily do-able.

1.	$\forall x(Fx \rightarrow \forall yGy)$	Pr
2.	$\sim \exists x(Gx \ \& \ \sim Fx)$	Pr
3.	SHOW: $\forall x(Fx \leftrightarrow Gx)$	UD
4.	SHOW: $Fa \leftrightarrow Ga$	DD
5.	SHOW: $Fa \rightarrow Ga$	CD
6.	Fa	Ass
7.	SHOW: Ga	ID
8.	$\sim Ga$	Ass
9.	SHOW: \ast	DD
10.	$\forall x \sim(Gx \rightarrow \sim Fx)$	2 $\sim \exists O$
11.	$Fa \rightarrow \forall yGy$	1 $\forall O$
12.	$\sim(Ga \rightarrow \sim Fa)$	10 $\forall O$
13.	$\forall yGy$	6,11 $\rightarrow O$
14.	Ga	13 $\forall O$
15.	\ast	8,14 $\ast I$
16.	SHOW: $Ga \rightarrow Fa$	CD
17.	Ga	Ass
18.	SHOW: Fa	ID
19.	$\sim Fa$	Ass
20.	SHOW: \ast	DD
21.	$\forall x \sim(Gx \rightarrow \sim Fx)$	2 $\sim \exists O$
22.	$Fa \rightarrow \forall yGy$	1 $\forall O$
23.	$\sim(Ga \rightarrow \sim Fa)$	10 $\forall O$
24.	$Ga \ \& \ \sim \sim Fa$	23 $\sim \rightarrow O$
25.	$\sim \sim Fa$	24 $\& O$
26.	\ast	19,25 $\ast I$
27.	$Fa \leftrightarrow Ga$	5,16 $\leftrightarrow I$

Notice that for a biconditional, we have to cycle through the steps independently for the two halves of the problem.

1.	$\exists x \sim Fx \rightarrow \exists x \sim Gx$	Pr
2.	$\forall x(Fx \rightarrow Gx)$	Pr
3.	SHOW: $\forall x Fx \leftrightarrow \forall x Gx$	DD
4.	SHOW: $\forall x Fx \rightarrow \forall x Gx$	CD
5.	$\forall x Fx$	Ass
6.	SHOW: $\forall x Gx$	UD
7.	SHOW: Ga	ID
8.	$\sim Ga$	Ass
9.	SHOW: \ast	DD
10.	Fa	5 $\forall O$
11.	$Fa \rightarrow Ga$	2 $\forall O$
12.	Ga	10,11 $\rightarrow O$
13.	\ast	8,12 $\ast I$
14.	SHOW: $\forall x Gx \rightarrow \forall x Fx$	CD
15.	$\forall x Gx$	Ass
16.	SHOW: $\forall x Fx$	UD
17.	SHOW: Fb	ID

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18.				~Fb	Ass
19.				SHOW: *	DD
20.				Gb	15 vO
21.				∃x~Fx	18 ∃I
22.				∃x~Gx	1,21 →O
23.				~Gc	22 ∃O
24.				Gc	15 vO
25.				*	23,24 *I
26.				∇xFx ↔ ∇xGx	4,13 ↔I

“Desperate Measures” Revisited

This is review from something we learned at the end of unit 2.

- If you get stuck in a proof, look at your *if-then* and *or* statements. What you need is a way to use →O or vO. To do this, you may need to introduce a new SHOW: line to try to prove either that one half of an *or* statement is false, or that the *if-part* of an *if-then* is true, or the *then-part* is false.

- Here are some example problems that require “desperate measures”.

1.				∃xRxx v ∃xFx	Pr
2.				SHOW: ∇x~Fx → ∃x∃yRxy	CD
3.				∇x~Fx	Ass
4.				SHOW: ∃x∃yRxy	ID
5.				~∃x∃yRxy	Ass
6.				SHOW: *	DD
7.				∇x~∃yRxy	5 ~∃O
8.				SHOW: ~∃xFx	ID
9.				∃xFx	Ass
10.				SHOW: *	DD
11.				Fa	9 ∃O
12.				~Fa	3 vO
13.				*	11,12 *I
14.				∃xRxx	1,8 vO
15.				Rbb	13 ∃O
16.				~∃yRby	7 vO
17.				∇y~Rby	16 ~∃O
18.				~Rbb	17 vO
19.				*	15,18 *I

The problem is set up normally. We get stuck at line 7, so we introduce a SHOW: line to give us what we would need to do vO on line 1. Once SHOW: is crossed off at line 8, we can use it.

1.				~∇x(Fx & Hx) → ∇x(Gx & ~Hx)	Pr
2.				SHOW: ∇xFx v ∇xGx	ID
3.				~(∇xFx v ∇xGx)	Ass
4.				SHOW: *	DD
5.				~∇xFx	3 ~vO
6.				~∇xGx	3 ~vO
7.				∃x~Fx	5 ~vO
8.				∃x~Gx	6 ~vO
9.				~Fa	7 ∃O
10.				~Gb	8 ∃O
11.				SHOW: ~∇x(Fx & Hx)	ID
12.				∇x(Fx & Hx)	Ass
13.				SHOW: *	DD
14.				Fa & Ha	12 vO
15.				Fa	14 &O
16.				*	9,15 *I
17.				∇x(Gx & ~Hx)	1,11 →O
18.				Gb & ~Hb	17 vO
19.				Gb	18 &O
20.				*	10,19 *I

1.				∇x(Fx → Gx) v ∃x~Hx	Pr
2.				∃x(~Gx & Fx)	Pr
3.				SHOW: ~∇xHx	ID
4.				∇xHx	Ass
5.				SHOW: *	DD
6.				~Ga & Fa	2 ∃O
7.				~Ga	6 &O
8.				Fa	6 &O
9.				SHOW: ~∃x~Hx	ID
10.				∃x~Hx	Ass
11.				SHOW: *	DD
12.				~Hb	10 ∃O
13.				Hb	4 vO
14.				*	12,13 *I
15.				∇x(Fx → Gx)	1,9 vO
16.				Fa → Ga	15 vO
17.				Ga	8,16 →O
18.				*	7,17 *I

Review Problems

1.				∃xRxx	Pr
2.				∇x∇y(Rxy ↔ Syx)	Pr
3.				SHOW: ~∇y∇x~Sxy	ID
4.				∇y∇x~Sxy	Ass
5.				SHOW: *	DD
6.				Raa	1 ∃O
7.				∇y(Ray ↔ Sya)	2 vO
8.				Raa ↔ Saa	7 vO

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9.	$\forall x \sim Sxa$	4 $\forall O$	1.	$\forall x(Fx \rightarrow \exists yGy)$	Pr
10.	$\sim Saa$	9 $\forall O$	2.	$\forall x(Gx \leftrightarrow \forall y \sim Fy)$	Pr
11.	$Raa \rightarrow Saa$	8 $\leftrightarrow O$	3.	SHOW: $\sim \exists xFx \ \& \ \forall xGx$	DD
12.	Saa	6,11 $\rightarrow O$	4.	SHOW: $\sim \exists xFx$	ID
13.	*	10,12 $\times I$	5.	$\exists xFx$	Ass
1.	$\exists x[Fx \ \& \ \forall y(Hy \rightarrow Rxy)]$	Pr	6.	SHOW: *	DD
2.	$\exists x(Gx \ \& \ Hx)$	Pr	7.	Fa	5 $\exists O$
3.	SHOW: $\exists x[Gx \ \& \ \sim \forall y(Fy \rightarrow \sim Ryx)]$	ID	8.	$Fa \rightarrow \exists yGy$	1 $\forall O$
4.	$\sim \exists x[Gx \ \& \ \sim \forall y(Fy \rightarrow \sim Ryx)]$	Ass	9.	$\exists yGy$	7,8 $\rightarrow O$
5.	SHOW: *	DD	10.	Gb	9 $\exists O$
6.	$\forall x \sim [Gx \ \& \ \sim \forall y(Fy \rightarrow \sim Ryx)]$	4 $\sim \exists O$	11.	$Gb \leftrightarrow \forall y \sim Fy$	2 $\forall O$
7.	$Fa \ \& \ \forall y(Hy \rightarrow Ray)$	1 $\exists O$	12.	$Gb \rightarrow \forall y \sim Fy$	11 $\leftrightarrow O$
8.	Gb & Hb	2 $\exists O$	13.	$\forall y \sim Fy$	10,12 $\rightarrow O$
9.	Fa	7 $\& O$	14.	$\sim Fa$	13 $\forall O$
10.	$\forall y(Hy \rightarrow Ray)$	7 $\& O$	15.	*	7,14 $\times I$
11.	Gb	8 $\& O$	16.	SHOW: $\forall xGx$	UD
12.	Hb	8 $\& O$	17.	SHOW: Gc	ID
13.	$Hb \rightarrow Rab$	10 $\forall O$	18.	$\sim Gc$	Ass
14.	Rab	12,13 $\rightarrow O$	19.	SHOW: *	DD
15.	$\sim [Gb \ \& \ \sim \forall y(Fy \rightarrow \sim Ryb)]$	6 $\forall O$	20.	$Gc \leftrightarrow \forall y \sim Fy$	2 $\forall O$
16.	$Gb \rightarrow \sim \sim \forall y(Fy \rightarrow \sim Ryb)$	15 $\sim \& O$	21.	$\forall y \sim Fy \rightarrow Gc$	20 $\leftrightarrow O$
17.	$\sim \sim \forall y(Fy \rightarrow \sim Ryb)$	11,16 $\rightarrow O$	22.	$\sim \forall y \sim Fy$	18,21 $\rightarrow O$
18.	$\forall y(Fy \rightarrow \sim Ryb)$	17 DN	23.	$\exists y \sim \sim Fy$	22 $\sim \forall O$
19.	$Fa \rightarrow \sim Rab$	18 $\forall O$	24.	$\sim \sim Fd$	23 $\exists O$
20.	$\sim Rab$	9,19 $\rightarrow O$	25.	Fd	24 DN
21.	*	14,20 $\times I$	26.	$\exists xFx$	25 $\exists I$
1.	$\exists y \forall x Ryx$	Pr	27.	*	4,26 $\times I$
2.	$\forall x(\exists y Ryx \rightarrow Rxx)$	Pr	28.	$\sim \exists xFx \ \& \ \forall xGx$	4,16 $\& I$
3.	SHOW: $\forall x Rxx$	UD	1.	$\forall x[(Fx \rightarrow Hx) \rightarrow (Gx \ \& \ Ix)]$	Pr
4.	SHOW: Raa	DD	2.	SHOW: $\forall x Hx \rightarrow \forall y Gy$	CD
5.	$\forall x Rbx$	1 $\exists O$	3.	$\forall x Hx$	Ass
6.	Rba	5 $\forall O$	4.	SHOW: $\forall y Gy$	UD
7.	$\exists y Rya \rightarrow Raa$	2 $\forall O$	5.	SHOW: Ga	ID
8.	$\exists y Rya$	6 $\exists I$	6.	$\sim Ga$	Ass
9.	Raa	7,8 $\rightarrow O$	7.	SHOW: *	DD
1.	$\exists x(Fx \vee Gx) \rightarrow \forall x Hx$	Pr	8.	Ha	3 $\forall O$
2.	SHOW: $\forall x \forall y(Gx \rightarrow Hy)$	UD	9.	$(Fa \rightarrow Ha) \rightarrow (Ga \ \& \ Ia)$	1 $\forall O$
3.	SHOW: $\forall y(Ga \rightarrow Hy)$	UD	10.	SHOW: $Fa \rightarrow Ha$	CD
4.	SHOW: $Ga \rightarrow Hb$	CD	11.	Fa	Ass
5.	Ga	Ass	12.	SHOW: Ha	DD
6.	SHOW: Hb	DD	13.	Ha	8 R
7.	Fa \vee Ga 5	$\vee I$	14.	Ga & Ia	9,10 $\rightarrow O$
8.	$\exists x(Fx \vee Gx)$	7 $\exists I$	15.	Ga	14 $\& O$
9.	$\forall x Hx$	1,8 $\rightarrow O$	16.	*	6,15 $\times I$
10.	Hb	9 $\forall O$			

And that's all for the semester!

Time to dance!