5 DERIVATIONS IN SENTENTIAL LOGIC

1. INTRODUCTION

In an earlier chapter, we studied a method of deciding whether an argument form of sentential logic is valid or invalid – the method of truth-tables. Although this method is infallible (when applied correctly), in many instances it can be tedious.

For example, if an argument form involves five distinct atomic formulas (say, P, Q, R, S, T), then the associated truth table contains 32 rows. Indeed, every additional atomic formula doubles the size of the associated truth-table. This makes the truth-table method impractical in many cases, unless one has access to a computer. Even then, due to the "doubling" phenomenon, there are argument forms that even a very fast main-frame computer cannot solve, at least in a reasonable amount of time (say, less than 100 years!)

Another shortcoming of the truth-table method is that it does not require much in the way of reasoning. It is simply a matter of mechanically following a simple set of directions. Accordingly, this method does not afford much practice in reasoning, either formal or informal.

For these two reasons, we now examine a second technique for demonstrating the validity of arguments – the method of *formal derivation*, or simply *derivation*. Not only is this method less tedious and mechanical than the method of truth tables, it also provides practice in symbolic reasoning.

Skill in symbolic reasoning can in turn be transferred to skill in practical reasoning, although the transfer is not direct. By analogy, skill in any game of strategy (say, chess) can be transferred indirectly to skill in general strategy (such as war, political or corporate). Of course, chess does not apply directly to any real strategic situation.

Constructing a derivation requires more thinking than filling out truth-tables. Indeed, in some instances, constructing a derivation demands considerable ingenuity, just like a good combination in chess.

Unfortunately, the method of formal derivation has its own shortcoming: unlike truth-tables, which can show both validity and invalidity, derivations can only show validity. If one succeeds in constructing a derivation, then one knows that the corresponding argument is valid. However, if one fails to construct a derivation, it does not mean that the argument is invalid. In the past, humans repeatedly failed to fly; this did not mean that flight was impossible. On the other hand, humans have repeatedly tried to construct perpetual motion machines, and they have failed. Sometimes failure is due to lack of cleverness; sometimes failure is due to the impossibility of the task!

2. THE BASIC IDEA

Underlying the method of formal derivations is the following fundamental idea.

> Granting the validity of a few selected argument forms, we can demonstrate the validity of other argument forms.

A simple illustration of this procedure might be useful. In an earlier chapter, we used the method of truth-tables to demonstrate the validity of numerous arguments. Among these, a few stand out for special mention. The first, and simplest one perhaps, is the following.

(MP) $P \rightarrow Q$
P
———
Q

This argument form is traditionally called *modus ponens*, which is short for *modus ponendo ponens*, which is a Latin expression meaning the mode of affirming by affirming. It is so called because, in this mode of reasoning, one goes from an affirmative premise to an affirmative conclusion.

It is easy to show that (MP) is a valid argument, using truth-tables. But we can use it to show other argument forms are also valid. Let us consider a simple example.

(a1) P
$P \rightarrow Q$
$Q \rightarrow R$
———
R

We can, of course, use truth-tables to show that (a1) is valid. Since there are three atomic formulas, 8 cases must be considered. However, we can also convince ourselves that (a1) is valid by reasoning as follows.

> Proof: Suppose the premises are all true. Then, in particular, the first two premises are both true. But if P and P→Q are both true, then Q must be true. Why? Because Q follows from P and P→Q by *modus ponens*. So now we know that the following formulas are all true: P, P→Q, Q, Q→R. This means that, in particular, both Q and Q→R are true. But R follows from Q and Q→R, by *modus ponens*, so R (the conclusion) must also be true. Thus, if the premises are all true, then so is the conclusion. In other words, the argument form is valid.

What we have done is show that (a1) is valid assuming that (MP) is valid.

Another important classical argument form is the following.

(MT)

$$\begin{array}{l} P \rightarrow Q \\ \sim Q \\ \hline \sim P \end{array}$$

This argument form is traditionally called *modus tollens*, which is short for *modus tollendo tollens*, which is a Latin expression meaning the mode of denying by denying. It is so called because, in this mode of reasoning, one goes from a negative premise to a negative conclusion.

Granting (MT), we can show that the following argument form is also valid.

(a2)

$$\begin{array}{l} P \rightarrow Q \\ Q \rightarrow R \\ \sim R \\ \hline \sim P \end{array}$$

Once again, we can construct a truth-table for (a2), which involves 8 lines. But we can also demonstrate its validity by the following reasoning.

> Proof: Suppose that the premises are all true. Then, in particular, the last two premises are both true. But if Q→R and ~R are both true, then ~Q is also true. For ~Q follows from Q→R and ~R, in virtue of *modus tollens*. So, if the premises are all true, then so is ~Q. That means that all the following formulas are true – P→Q, Q→R, ~R, ~Q. So, in particular, P→Q and ~Q are both true. But if these are true, then so is ~P (the conclusion), because ~P follows from P→Q and ~Q, in virtue of *modus tollens*. Thus, if the premises are all true, then so is the conclusion. In other words, the argument form is valid.

Finally, let us consider an example of reasoning that appeals to both *modus ponens* and *modus tollens*.

(a3)

$$\begin{array}{l} \sim P \\ \sim P \rightarrow \sim R \\ Q \rightarrow R \\ \hline \sim Q \end{array}$$

Proof: Suppose that the premises are all true. Then, in particular, the first two premises are both true. But if ~P and ~P→~R are both true, then so is ~R, in virtue of *modus ponens*. Then ~R and Q→R are both true, but then ~Q is true, in virtue of *modus tollens*. Thus, if the premises are all true, then the conclusion is also true, which is to say the argument is valid.

3. ARGUMENT FORMS AND SUBSTITUTION INSTANCES

In the previous section, the alert reader probably noticed a slight discrepancy between the official argument forms (MP) and (MT), on the one hand, and the actual argument forms appearing in the proofs of the validity of (a1)-(a3).

For example, in the proof of (a3), I said that ~R follows from ~P and ~P→~R, in virtue of *modus ponens*. Yet the argument forms are quite different.

(MP)
P → Q
P
———
Q

(MP*)
~P → ~R
~P
————
~R

(MP*) looks somewhat like (MP); if we squinted hard enough, we might say they looked the same. But, clearly, (MP*) is not exactly the same as (MP). In particular, (MP) has no occurrences of negation, whereas (MP*) has 4 occurrences. So, in what sense can I say that (MP*) is valid *in virtue of* (MP)?

The intuitive idea is that "the overall form" of (MP*) is the same as (MP). (MP*) is an argument form with the following overall form.

conditional formula	() → []
antecedent	()
————————	———
consequent	[]

The fairly imprecise notion of overall form can be made more precise by appealing to the notion of a substitution instance. We have already discussed this notion earlier. The slight complication here is that, rather than substituting a concrete argument for an argument form, we substitute one argument form for another argument form,

The following is the official definition.

> **Definition:**
> If ***A*** is an argument form of sentential logic, then a *substitution instance* of ***A*** is any argument form ***A**** that is obtained from ***A*** by substituting formulas for letters in ***A***.

There is an affiliated definition for formulas.

> **Definition:**
> If ***F*** is a formula of sentential logic, then a *substitution instance* of ***F*** is any formula ***F**** obtained from ***F*** by substituting formulas for letters in ***F***.

Note carefully: it is understood here that if a formula replaces a given letter in one place, then the formula replaces the letter in every place. One cannot substitute different formulas for the same letter. However, one is permitted to replace two different letters by the same formula. This gives rise to the notion of uniform substitution instance.

> **Definition:**
> A substitution instance is a *uniform substitution instance* if and only if distinct letters are replaced by distinct formulas.

These definitions are best understood in terms of specific examples. First, (MP*) is a (uniform) substitution of (MP), obtained by substituting ~P for P, and ~R for Q. The following are examples of substitution instances of (MP)

~P → ~Q	(P & Q) → ~R	(P → Q) → (P → R)
~P	P & Q	P → Q
~Q	~R	P → R

Whereas (MP*) is a substitution instance of (MP), the converse is not true: (MP) is not a substitution instance of (MP*). There is no way to substitute formulas for letters in (MP*) in such a way that (MP) is the result. (MP*) has four negations, and (MP) has none. A substitution instance F* always has at least as many occurrences of a connective as the original form F.

The following are substitution instances of (MP*).

~(P & Q) → ~(P → Q)	~~P → ~(Q ∨ R)
~(P & Q)	~~P
~(P → Q)	~(Q ∨ R)

Interestingly enough these are also substitution instances of (MP). Indeed, we have the following general theorem.

> **Theorem:**
> If argument form ***A**** is a substitution instance of ***A***, and argument form ***A***** is a substitution instance of ***A****, then ***A***** is a substitution instance of ***A***.

With the notion of substitution instance in hand, we are now in a position to solve the original problem. To say that argument form (MP*) is valid *in virtue of* modus ponens (MP) is not to say that (MP*) is identical to (MP); rather, it is to say that (MP*) is a substitution instance of (MP). The remaining question is whether the validity of (MP) ensures the validity of its substitution instances. This is answered by the following theorem.

> **Theorem:**
> If argument form ***A*** is valid,
> then every substitution instance of ***A*** is also valid.

The *rigorous* proof of this theorem is beyond the scope of introductory logic.

4. SIMPLE INFERENCE RULES

In the present section, we lay down the ground work for constructing our system of formal derivation, which we will call system SL (short for 'sentential logic'). At the heart of any derivation system is a set of *inference rules*. Each inference rule corresponds to a valid argument of sentential logic, although not every valid argument yields a corresponding inference rule. We select a subset of valid arguments to serve as inference rules.

But how do we make the selection? On the one hand, we want to be parsimonious. We want to employ as few inference rules as possible and still be able to generate all the valid argument forms. On the other hand, we want each inference rule to be simple, easy to remember, and intuitively obvious. These two desiderata actually push in opposite directions; the most parsimonious system is not the most intuitively clear; the most intuitively clear system is not the most parsimonious. Our particular choice will accordingly be a compromise solution.

We have to select from the infinitely-many valid argument forms of sentential logic a handful of very fertile ones, ones that will generate the rest. To a certain extent, the choice is arbitrary. It is very much like inventing a game – we get to make up the rules. On the other hand, the rules are not entirely arbitrary, because each rule must correspond to a valid argument form. Also, note that, even though we can choose the rules initially, once we have chosen, we must adhere to the ones we have chosen.

Every inference rule corresponds to a valid argument form of sentential logic. Note, however, that in granting the validity of an argument form (say, *modus ponens*), we mean to grant that specific argument form as well as *every* substitution instance.

In order to convey that each inference rule subsumes infinitely many argument forms, we will use an alternate font to formulate the inference rules; in particular, capital script letters ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc.) will stand for arbitrary formulas of sentential logic.

Thus, for example, the rule of *modus ponens* will be written as follows, where $\mathcal{A}$ and $\mathcal{C}$ are *arbitrary* formulas of sentential logic.

(MP)
$$\begin{array}{l} \mathcal{A} \rightarrow \mathcal{C} \\ \mathcal{A} \\ \hline \mathcal{C} \end{array}$$

Given that the script letters '$\mathcal{A}$' and '$\mathcal{C}$' stand for arbitrary formulas, (MP) stands for infinitely many argument forms, all looking like the following.

(MP)

	conditional	(antecedent) → [consequent]
	antecedent	(antecedent)
	———	———
	consequent	[consequent]

Along the same lines, the rule *modus tollens* may be written as follows.

(MT)
$$\begin{array}{l} \mathcal{A} \rightarrow \mathcal{C} \\ \sim\mathcal{C} \\ \hline \sim\mathcal{A} \end{array}$$

(MT)

	conditional	(antecedent) → [consequent]
	literal negation of consequent	~[consequent]
	———	———
	literal negation of antecedent	~(antecedent)

Note: By 'literal negation of formula $\mathcal{A}$' is meant the formula that results from prefixing the formula $\mathcal{A}$ with a tilde. The literal negation of a formula always has exactly one more symbol than the formula itself.

In addition to (MP) and (MT), there are two other similar rules that we are going to adopt, given as follows.

(MTP1)
$$\begin{array}{l} \mathcal{A} \vee \mathcal{B} \\ \sim\mathcal{A} \\ \hline \mathcal{B} \end{array}$$

(MTP2)
$$\begin{array}{l} \mathcal{A} \vee \mathcal{B} \\ \sim\mathcal{B} \\ \hline \mathcal{A} \end{array}$$

This mode of reasoning is traditionally called *modus tollendo ponens*, which means the mode of affirming by denying. In each case, an affirmative conclusion is reached on the basis of a negative premise. The reader should verify, using truth-tables, that the simplest instances of these inference rules are in fact valid. The reader should also verify the intuitive validity of these forms of reasoning. MTP corresponds to the "process of elimination": one has a choice between two things, one eliminates one choice, leaving the other.

Before putting these four rules to work, it is important to point out two classes of errors that a student is liable to make.

• Errors of the First Kind

The four rules given above are to be carefully distinguished from argument forms that look similar but are clearly invalid. The following arguments are not instances of any of the above rules; worse, they are invalid.

Invalid!	Invalid!	Invalid!	Invalid!
P → Q Q ——— P	P → Q ~P ——— ~Q	P ∨ Q P ——— ~Q	P ∨ Q Q ——— ~P

These modes of inference are collectively known as *modus morons*, which means the mode of reasoning like a moron. It is easy to show that every one of them is invalid. You can use truth-tables, or you can construct counter-examples; either way, they are invalid.

• Errors of the Second Kind

Many valid arguments are not substitution instances of inference rules. This isn't too surprising. Some arguments, however, *look like* (but are not) substitution instances of inference rules. The following are examples.

Valid but not MT!	Valid but not MT!	Valid but not MTP!	Valid but not MTP!
~P → Q ~Q ——— P	P → ~Q Q ——— ~P	~P ∨ ~Q P ——— ~Q	~P ∨ ~Q Q ——— ~P

The following are corresponding correct applications of the rules.

MT	MT	MTP	MTP
~P → Q ~Q ——— ~~P	P → ~Q ~~Q ——— ~P	~P ∨ ~Q ~~P ——— ~Q	~P ∨ ~Q ~~Q ——— ~P

The natural question is, "aren't ~~P and P the same?" In asking this question, one might be thinking of arithmetic: for example, --2 and 2 are one and same *number*. But the corresponding *numerals* are not identical: the linguistic expression '--2' is not identical to the linguistic expression '2'. Similarly, the Roman *numeral* 'VII' is not identical to the Arabic *numeral* '7' even though both *numerals* denote the same *number*. Just like people, numbers have names; the names of numbers are numerals. We don't confuse people and their names. We shouldn't confuse numbers and their names (numerals).

Thus, the answer is that the formulas ~~P and P are not the same; they are as different as the Roman numeral 'VII' and the Arabic numeral '7'.

Another possible reason to think ∼∼P and P are the same is that they are *logically equivalent*, which may be shown using truth tables. This means they have the same truth-value no matter what. They have the same truth-value; does that mean they are the same? Of course not! That is like arguing from the premise that John and Mary are legally equivalent (meaning that they are equal under the law) to the conclusion that John and Mary are the same. Logical equivalence, like legal equivalence, is not identity.

Consider a very similar question whose answer revolves around the distinction between equality and identity: are four quarters and a dollar bill the same? The answer is, "yes and no". Four quarters are *monetarily equal* to a dollar bill, but they are definitely not identical. Quarters are made of metal, dollar bills are made of paper; they are physically quite different. For some purposes they are interchangeable; that does not mean they are the same.

The same can be said about ∼∼P and P. They have the same *value* (in the sense of truth-value), but they are definitely not identical. One has three symbols, the other only one, so they are not identical. More importantly, for our purposes, they have different forms – one is a negation; the other is atomic.

A derivation system in general, and inference rules in particular, pertain exclusively to the forms of the formulas involved.

In this respect, derivation systems are similar to coin-operated machines – vending machines, pay phones, parking meters, automatic toll booths, etc. A vending machine, for example, does not "care" what the *value* of a coin is. It only "cares" about the coin's form; it responds exclusively to the shape and weight of the coin. A penny worth one dollar to collectors won't buy a soft drink from a vending machine. Similarly, if the machine does not accept pennies, it is no use to put in 25 of them, even though 25 pennies have the same monetary value as a quarter. Similarly frustrating at times, a dollar bill is worthless when dealing with many coin-operated machines.

A derivation system is equally "stubborn"; it is blind to content, and responds exclusively to form. The fact that truth-tables tell us that P and ∼∼P are logically equivalent is irrelevant. If P is required by an inference-rule, then ∼∼P won't work, and if ∼∼P is required, then P won't work, just like 25 pennies won't buy a stick of gum from a vending machine. What one must do is first trade P for ∼∼P. We will have such conversion rules available.

5. SIMPLE DERIVATIONS

We now have four inference rules, MP, MT, MTP1, and MTP2. How do we utilize these in demonstrating other arguments of sentential logic are also valid? In order to prove (show, demonstrate) that an argument is valid, one derives its conclusion from its premises. We have already seen intuitive examples in an earlier section. We now redo these examples formally.

The first technique of derivation that we examine is called *simple derivation*. It is temporary, and will be replaced in the next section. However, it demonstrates the key intuitions about derivations.

Simple derivations are defined as follows.

> **Definition:**
> A *simple derivation* of conclusion $\mathcal{C}$ from premises $\mathcal{P}_1$, $\mathcal{P}_2$, ..., $\mathcal{P}_n$ is a list of formulas (also called lines) satisfying the following conditions.
>
> (1) the last line is $\mathcal{C}$;
> (2) every line (formula) is
>
> either:
> a premise (one of $\mathcal{P}_1$, $\mathcal{P}_2$, ..., $\mathcal{P}_n$),
> or:
> follows from previous lines according to an inference rule.

The basic idea is that in order to prove that an argument is valid, it is sufficient to construct a simple derivation of its conclusion from its premises. Rather than dwell on abstract matters of definition, it is better to deal with some examples by way of explaining the method of simple derivation.

Example 1

Argument: $P ; P \rightarrow Q ; Q \rightarrow R / R$

Simple Derivation:

(1)	P	Pr
(2)	$P \rightarrow Q$	Pr
(3)	$Q \rightarrow R$	Pr
(4)	Q	1,2,MP
(5)	R	3,4,MP

This is an example of a simple derivation. The last line is the conclusion; every line is either a premise or follows by a rule. The annotation to the right of each formula indicates the precise justification for the presence of the formula in the derivation. There are two possible justifications at the moment; the formula is a premise (annotation: 'Pr'); the formula follows from previous formulas by a rule (annotation: line numbers, rule).

Example 2

Argument: $P \to Q$; $Q \to R$; $\sim R$ / $\sim P$

Simple Derivation:

(1)	$P \to Q$	Pr
(2)	$Q \to R$	Pr
(3)	$\sim R$	Pr
(4)	$\sim Q$	2,3,MT
(5)	$\sim P$	1,4,MT

Example 3

Argument: $\sim P$; $\sim P \to \sim R$; $Q \to R$ / $\sim Q$

Simple Derivation:

(1)	$\sim P$	Pr
(2)	$\sim P \to \sim R$	Pr
(3)	$Q \to R$	Pr
(4)	$\sim R$	1,2,MP
(5)	$\sim Q$	3,4,MT

These three examples take care of the examples from Section 2. The following one is more unusual.

Example 4

Argument: $(P \to Q) \to P$; $P \to Q$ / Q

Simple Derivation:

(1)	$(P \to Q) \to P$	Pr
(2)	$P \to Q$	Pr
(3)	P	1,2,MP
(4)	Q	2,3,MP

What is unusual about this one is that line (2) is used twice, in connection with MP, once as minor premise, once as major premise. One can appeal to the same line over and over again, if the need arises.

We conclude this section with examples of slightly longer simple derivations.

Example 5

Argument: P → (Q ∨ R) ; P → ~R ; P / Q

Simple Derivation:

(1)	P → (Q ∨ R)	Pr
(2)	P → ~R	Pr
(3)	P	Pr
(4)	~R	2,3,MP
(5)	Q ∨ R	1,3,MP
(6)	Q	4,5,MTP2

Example 6

Argument: ~P → (Q ∨ R) ; P → Q ; ~Q / R

Simple Derivation:

(1)	~P → (Q ∨ R)	Pr
(2)	P → Q	Pr
(3)	~Q	Pr
(4)	~P	2,3,MT
(5)	Q ∨ R	1,4,MP
(6)	R	3,5,MTP1

Example 7

Argument: (P ∨ R) ∨ (P → Q) ; ~(P → Q) ; R → (P → Q) / P

Simple Derivation:

(1)	(P ∨ R) ∨ (P → Q)	Pr
(2)	~(P → Q)	Pr
(3)	R → (P → Q)	Pr
(4)	P ∨ R	1,2,MTP2
(5)	~R	2,3,MT
(6)	P	4,5,MTP2

Example 8

Argument: P → ~Q ; ~Q → (R & S) ; ~(R & S) ; P ∨ T / T

Simple Derivation:

(1)	P → ~Q	Pr
(2)	~Q → (R & S)	Pr
(3)	~(R & S)	Pr
(4)	P ∨ T	Pr
(5)	~~Q	2,3,MT
(6)	~P	1,5,MT
(7)	T	4,6,MTP1

6. THE OFFICIAL INFERENCE RULES

So far, we have discussed only four inference rules: *modus ponens*, *modus tollens*, and the two forms of *modus tollendo ponens*. In the present section, we add quite a few more inference rules to our list.

Since the new rules will be given more pictorial, non-Latin, names, we are going to rename our original four rules in order to maintain consistency. Also, we are going to consolidate our original four rules into two rules.

In constructing the full set of inference rules, we would like to pursue the following overall plan. For each of the five connectives, we want two rules: on the one hand, we want a rule for "introducing" the connective; on the other hand, we want a rule for "eliminating" the connective. An introduction-rule is also called an *in-rule*; an elimination-rule is called an *out-rule*.

Also, it would be nice if the name of each rule is suggestive of what the rule does. In particular, the name should consist of two parts: (1) reference to the specific connective involved, and (2) indication whether the rule is an introduction (in) rule or an elimination (out) rule.

Thus, if we were to follow the overall plan, we would have a total of ten rules, listed as follows.

Ampersand-In	&I
Ampersand-Out	&O
Wedge-In	∨I
Wedge-Out	∨O
Double-Arrow-In	↔I
Double-Arrow-Out	↔O
*Arrow-In	→I
Arrow-Out	→O
*Tilde-In	~I
*Tilde-Out	~O

However, for reasons of simplicity of presentation, the general plan is not followed completely. In particular, there are three points of difference, which are marked by an asterisk. What we adopt instead, in the derivation system SL, are the following inference rules.

7. INFERENCE RULES (INITIAL SET)

Rule		
Ampersand-In (&I)	$\mathcal{A}$ $\mathcal{B}$ ――――― $\mathcal{A} \,\&\, \mathcal{B}$	$\mathcal{A}$ $\mathcal{B}$ ――――― $\mathcal{B} \,\&\, \mathcal{A}$
Ampersand-Out (&O)	$\mathcal{A} \,\&\, \mathcal{B}$ ――――― $\mathcal{A}$	$\mathcal{A} \,\&\, \mathcal{B}$ ――――― $\mathcal{B}$
Wedge-In (∨I)	$\mathcal{A}$ ――――― $\mathcal{A} \vee \mathcal{B}$	$\mathcal{A}$ ――――― $\mathcal{B} \vee \mathcal{A}$
Wedge-Out (∨O)	$\mathcal{A} \vee \mathcal{B}$ $\sim\mathcal{A}$ ――――― $\mathcal{B}$	$\mathcal{A} \vee \mathcal{B}$ $\sim\mathcal{B}$ ――――― $\mathcal{A}$
Double-Arrow-In (↔I)	$\mathcal{A} \rightarrow \mathcal{B}$ $\mathcal{B} \rightarrow \mathcal{A}$ ――――― $\mathcal{A} \leftrightarrow \mathcal{B}$	$\mathcal{A} \rightarrow \mathcal{B}$ $\mathcal{B} \rightarrow \mathcal{A}$ ――――― $\mathcal{B} \leftrightarrow \mathcal{A}$
Double-Arrow-Out (↔O)	$\mathcal{A} \leftrightarrow \mathcal{B}$ ――――― $\mathcal{A} \rightarrow \mathcal{B}$	$\mathcal{A} \leftrightarrow \mathcal{B}$ ――――― $\mathcal{B} \rightarrow \mathcal{A}$
Arrow-Out (→O)	$\mathcal{A} \rightarrow \mathcal{B}$ $\mathcal{A}$ ――――― $\mathcal{B}$	$\mathcal{A} \rightarrow \mathcal{B}$ $\sim\mathcal{B}$ ――――― $\sim\mathcal{A}$
Double Negation (DN)	$\mathcal{A}$ ――――― $\sim\sim\mathcal{A}$	$\sim\sim\mathcal{A}$ ――――― $\mathcal{A}$

A few notes may help clarify the above inference rules.

Notes

(1) Arrow-out (→O), the rule for decomposing conditional formulas, replaces both *modus ponens* and *modus tollens*.

(2) Wedge-out (∨O), the rule for decomposing disjunctions, replaces both forms of *modus tollendo ponens*.

(3) Double negation (DN) stands in place of both the tilde-in and the tilde-out rule.

(4) There is *no* arrow-in rule! [The rule for introducing arrow is not an inference rule but rather a show-rule, which is a different kind of rule, to be discussed later.]

(5) In each of the rules, $\mathcal{A}$ and $\mathcal{B}$ are arbitrary formulas of sentential logic. Each rule is short for infinitely many substitution instances.

(6) In each of the rules, the order of the premises is completely irrelevant.

(7) In the wedge-in (∨I) rule, the formula $\mathcal{B}$ is *any* formula whatsoever; it does not even have to be anywhere near the derivation in question!

There is one point that is extremely important, given as follows, which will be repeated as the need arises.

> Inference rules apply
> to whole lines,
> not to pieces of lines.

In other words, what are given above are not actually the inference rules themselves, but only pictures suggestive of the rules. The actual rules are more properly written as follows.

8. INFERENCE RULES; OFFICIAL FORMULATION

> **Ampersand-In** (&I): If one has available *lines*, $\mathcal{A}$ and $\mathcal{B}$, then one is entitled to write down their conjunction, in one order $\mathcal{A}$&$\mathcal{B}$, or the other order $\mathcal{B}$&$\mathcal{A}$.

> **Ampersand-Out** (&O): If one has available a *line* of the form $\mathcal{A}$&$\mathcal{B}$, then one is entitled to write down either conjunct $\mathcal{A}$ or conjunct $\mathcal{B}$.

> **Wedge-In** (∨I): If one has available a *line* $\mathcal{A}$, then one is entitled to write down the disjunction of $\mathcal{A}$ with any formula $\mathcal{B}$, in one order $\mathcal{A}$v$\mathcal{B}$, or the other order $\mathcal{B}$v$\mathcal{A}$.

Wedge-Out (∨O): If one has available a *line* of the form $\mathcal{A} \vee \mathcal{B}$, and if one additionally has available a *line* which is the negation of the first disjunct, $\sim\mathcal{A}$, then one is entitled to write down the second disjunct, $\mathcal{B}$. Likewise, if one has available a *line* of the form $\mathcal{A} \vee \mathcal{B}$, and if one additionally has available a *line* which is the negation of the second disjunct, $\sim\mathcal{B}$, then one is entitled to write down the first disjunct, $\mathcal{A}$.

Double-Arrow-In (↔I): If one has available a *line* that is a conditional $\mathcal{A} \rightarrow \mathcal{B}$, and one additionally has available a line that is the converse $\mathcal{B} \rightarrow \mathcal{A}$, then one is entitled to write down either the biconditional $\mathcal{A} \leftrightarrow \mathcal{B}$ or the biconditional $\mathcal{B} \leftrightarrow \mathcal{A}$.

Double-Arrow-Out (↔O): If one has available a *line* of the form $\mathcal{A} \leftrightarrow \mathcal{B}$, then one is entitled to write down both the conditional $\mathcal{A} \rightarrow \mathcal{B}$ and its converse $\mathcal{B} \rightarrow \mathcal{A}$.

Arrow-Out (→O): If one has available a *line* of the form $\mathcal{A} \rightarrow \mathcal{B}$, and if one additionally has available a *line* which is the antecedent $\mathcal{A}$, then one is entitled to write down the consequent $\mathcal{B}$. Likewise, if one has available a *line* of the form $\mathcal{A} \rightarrow \mathcal{B}$, and if one additionally has available a *line* which is the negation of the consequent, $\sim\mathcal{B}$, then one is entitled to write down the negation of the antecedent, $\sim\mathcal{A}$.

Double Negation (DN): If one has available a *line* $\mathcal{A}$, then one is entitled to write down the double-negation $\sim\sim\mathcal{A}$. Similarly, if one has available a *line* of the form $\sim\sim\mathcal{A}$, then one is entitled to write down the formula $\mathcal{A}$.

The word ‘available’ is used in a technical sense that will be explained in a later section.

To this list, we will add a few further inference rules in a later section. They are not crucial to the derivation system; they merely make doing derivations more convenient.

9. SHOW-LINES AND SHOW-RULES; DIRECT DERIVATION

Having discussed simple derivations, we now begin the official presentation of the derivation system SL. In constructing system SL, we lay down a set of *system rules* – the rules of SL. It's a bit confusing: we have inference rules, already presented; now we have system rules as well. System rules are simply the official rules for constructing derivations, and include, among other things, all the inference rules.

For example, we have already seen two system rules, in effect. They are the two principles of simple derivation, which are now officially formulated as system rules.

System Rule 1 (The Premise Rule)

> At any point in a derivation, ***prior*** to the first show-line, any premise may be written down. The annotation is 'Pr'.

System Rule 2 (The Inference-Rule Rule)

> At any point in a derivation, a formula may be written down if it follows from previous available lines by an inference rule. The annotation cites the line numbers, and the inference rule, in that order.

System Rule 2 is actually short-hand for the list of all the inference rules, as formulated at the end of Section 6.

The next thing we do in elaborating system SL is to enhance the notion of simple derivation to obtain the notion of a direct derivation. This enhancement is quite simple; it even seems redundant, at the moment. But as we further elaborate system SL, this enhancement will become increasingly crucial. Specifically, we add the following additional system rule, which concerns a new kind of line, called a *show-line*, which may be introduced at any point in a derivation.

System Rule 3 (The Show-Line Rule)

> At *any* point in a derivation, one is entitled to write down the expression 'SHOW: $\mathcal{A}$',
> for *any* formula $\mathcal{A}$ *whatsoever*.

In writing down the line 'SHOW: $\mathcal{A}$', all one is saying is, "I will now *attempt* to show the formula $\mathcal{A}$". What the rule amounts to, then, is that at any point one is entitled to *attempt* to show anything one pleases. This is very much like saying that any citizen (over a certain age) is entitled to run for president. But rights are not guarantees; you can try, but you may not succeed.

Allowing show-lines changes the derivation system quite a bit, at least in the long run. However, at the current stage of development of system SL, there is generally only one reasonable kind of show-line. Specifically, one writes down 'SHOW: *C*', where *C* is the conclusion of the argument one is trying to prove valid. Later, we will see other uses of show-lines.

All derivations start pretty much the same way: one writes down all the premises, as permitted by System Rule 1; then one writes down 'SHOW: *C*' (where *C* is the conclusion), which is permitted by System Rule 3.

Consider the following example, which is the beginning of a derivation.

Example 1

(1)	(P ∨ Q) → ~R	Pr
(2)	P & T	Pr
(3)	R ∨ ~S	Pr
(4)	U → S	Pr
(5)	SHOW: ~U	???

These five lines may be regarded as simply stating the problem – we want to show one formula, given four others. I write '???' in the annotation column because this still needs explaining; more about this later.

Given the problem, we can construct what is very similar to a simple derivation, as follows.

(1)	(P ∨ Q) → ~R	Pr
(2)	P & T	Pr
(3)	R ∨ ~S	Pr
(4)	U → S	Pr
(5)	SHOW: ~U	???
(6)	P	2,&O
(7)	P ∨ Q	6,∨I
(8)	~R	1,7,→O
(9)	~S	3,8,∨O
(10)	~U	4,9,→O

Notice that, if we deleted the show-line, (5), the result is a simple derivation.

We are allowed to *try* to show anything. But how do we know when we have succeeded? In order to decide when a formula has in fact been shown, we need additional system rules, which we call "show-rules". The first show-rule is so simple it barely requires mentioning. Nevertheless, in order to make system SL completely clear and precise, we must make this rule explicit.

The first show-rule may be intuitively formulated as follows.

> **Direct Derivation** (Intuitive Formulation)
>
> If one is trying to show formula $\mathcal{A}$, and one actually obtains $\mathcal{A}$ as a later line, then one has succeeded.

The intuitive formulation is, unfortunately, not sufficiently precise for the purposes to which it will ultimately be put. So we formulate the following official system rule of derivation.

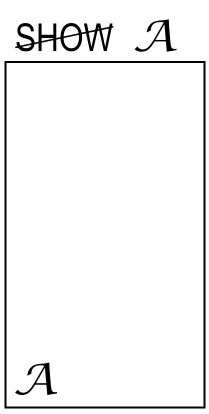
> **System Rule 4** (a show-rule)
>
> **Direct Derivation (DD)**
>
> If one has a show-line '`SHOW`: $\mathcal{A}$', and one obtains $\mathcal{A}$ as a later **available** line, and there are no intervening uncancelled show-lines, then one is entitled to box and cancel '`SHOW`: $\mathcal{A}$'. The annotation is 'DD'

As it is officially written, direct derivation is a very complicated rule. Don't worry about it now. The subtleties of the rule don't come into play until later.

For the moment, however, we do need to understand the idea of cancelling a show-line and boxing off the associated sub-derivation. Cancelling a show-line simply amounts to striking through the word '`SHOW`', to obtain '~~`SHOW`~~'. This indicates that the formula has in fact been shown. Now the formula $\mathcal{A}$ can be used. The trade-off is that one must box off the associated derivation. No line inside a box can be further used. One, in effect, trades the derivation for the formula shown. More about this restriction later.

The intuitive content of direct derivation is pictorially presented as follows.

Direct Derivation (DD)



The box is of little importance right now, but later it becomes very important in helping organize very complex derivations, ones that involve several show-lines. For the moment, simply think of the box as a decoration, a flourish if you like, to celebrate having shown the formula.

Let us return to our original derivation problem. Completing it according to the strict rules yields the following.

(1)	(P ∨ Q) → ~R	Pr
(2)	P & T	Pr
(3)	R ∨ ~S	Pr
(4)	U → S	Pr
(5)	~~SHOW~~: ~U	DD
(6)	P	2,&O
(7)	P ∨ Q	6,∨I
(8)	~R	1,7,→O
(9)	~S	3,8,∨O
(10)	~U	4,9,→O

Note that ‘SHOW’ has been struck through, resulting in ‘~~SHOW~~’. Note the annotation for line (5); ‘DD’ indicates that the show-line has been cancelled in accordance with the show-rule Direct Derivation. Finally, note that every formula below the show-line has been boxed off.

Later, we will have other, more complicated, show-rules. For the moment, however, we just have direct derivation.

10. EXAMPLES OF DIRECT DERIVATIONS

In the present section, we look at several examples of direct derivations.

Example 1

(1)	~P → (Q ∨ R)	Pr
(2)	P → Q	Pr
(3)	~Q	Pr
(4)	~~SHOW~~: R	DD
(5)	~P	2,3,→O
(6)	Q ∨ R	1,5,→O
(7)	R	3,6,∨O

Example 2

(1)	P & Q	Pr
(2)	~~SHOW~~: ~~P & ~~Q	DD
(3)	P	1,&O
(4)	Q	1,&O
(5)	~~P	3,DN
(6)	~~Q	4,DN
(7)	~~P & ~~Q	5,6,&I

Example 3

(1)	$P \& Q$	Pr
(2)	$(Q \vee R) \rightarrow S$	Pr
(3)	~~SHOW~~: $P \& S$	DD
(4)	P	1,&O
(5)	Q	1,&O
(6)	$Q \vee R$	5,$\vee$I
(7)	S	2,6,$\rightarrow$O
(8)	$P \& S$	4,7,&I

Example 4

(1)	$A \& B$	Pr
(2)	$(A \vee E) \rightarrow C$	Pr
(3)	$D \rightarrow {\sim}C$	Pr
(4)	~~SHOW~~: ${\sim}D$	DD
(5)	A	1,&O
(6)	$A \vee E$	5,$\vee$I
(7)	C	2,6,$\rightarrow$O
(8)	${\sim}{\sim}C$	7,DN
(9)	${\sim}D$	3,8,$\rightarrow$O

Example 5

(1)	$A \& {\sim}B$	Pr
(2)	$B \vee (A \rightarrow D)$	Pr
(3)	$(C \& E) \leftrightarrow D$	Pr
(4)	~~SHOW~~: $A \& C$	DD
(5)	A	1,&O
(6)	${\sim}B$	1,&O
(7)	$A \rightarrow D$	2,6,$\vee$O
(8)	D	5,7,$\rightarrow$O
(9)	$D \rightarrow (C \& E)$	3,$\leftrightarrow$O
(10)	$C \& E$	8,9,$\rightarrow$O
(11)	C	10,&O
(12)	$A \& C$	5,11,&I

Example 6

(1)	$A \rightarrow B$	Pr
(2)	$(A \rightarrow B) \rightarrow (B \rightarrow A)$	Pr
(3)	$(A \leftrightarrow B) \rightarrow A$	Pr
(4)	~~SHOW~~: $A \& B$	DD
(5)	$B \rightarrow A$	1,2,$\rightarrow$O
(6)	$A \leftrightarrow B$	1,5,$\leftrightarrow$I
(7)	A	3,6,$\rightarrow$O
(8)	B	1,7,$\rightarrow$O
(9)	$A \& B$	7,8,&I

Example 7

(1)	~A & B	Pr
(2)	(C ∨ B) → (~D → A)	Pr
(3)	~D ↔ E	Pr
(4)	~~SHOW~~: ~E	DD
(5)	~A	1,&O
(6)	B	1,&O
(7)	C ∨ B	6,∨I
(8)	~D → A	2,7,→O
(9)	~~D	5,8,→O
(10)	E → ~D	3,↔O
(11)	~E	9,10,→O

NOTE: From now on, for the sake of typographical neatness, we will draw boxes in a purely skeletal fashion. In particular, we will only draw the left side of each box; the remaining sides of each box should be mentally filled in. For example, using skeletal boxes, the last two derivations are written as follows.

Example 6 (rewritten)

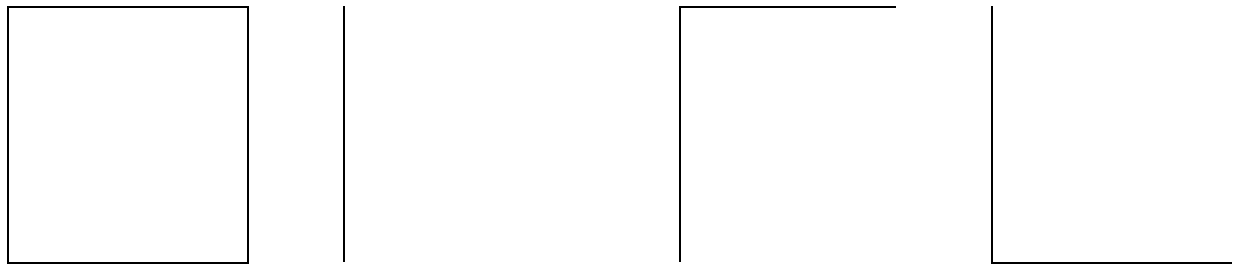
(1)	A → B	Pr
(2)	(A → B) → (B → A)	Pr
(3)	(A ↔ B) → A	Pr
(4)	~~SHOW~~: A & B	DD
(5)	B → A	1,2,→O
(6)	A ↔ B	1,5,↔I
(7)	A	3,6,→O
(8)	B	1,7,→O
(9)	A & B	7,8,&I

Example 7 (rewritten)

(1)	~A & B	Pr
(2)	(C ∨ B) → (~D → A)	Pr
(3)	~D ↔ E	Pr
(4)	~~SHOW~~: ~E	DD
(5)	~A	1,&O
(6)	B	1,&O
(7)	C ∨ B	6,∨I
(8)	~D → A	2,7,→O
(9)	~~D	5,8,→O
(10)	E → ~D	3,↔O
(11)	~E	9,10,→O

NOTE: In your own derivations, you can draw as much, or as little, of a box as you like, so long as you include at a minimum its left side. For example, you can use any of the following schemes.

~~SHOW~~: ~~SHOW~~: ~~SHOW~~: ~~SHOW~~:



Finally, we end this section by rewriting the Direct Derivation Picture, in accordance with our minimal boxing scheme.

Direct Derivation (DD)

~~SHOW~~: $\mathcal{A}$ DD
o
o
o
o
o
o
o
$\mathcal{A}$

11. CONDITIONAL DERIVATION

So far, we only have one method by which to cancel a show-line – direct derivation. In the present section, we examine a new derivation method, which will enable us to prove valid a larger class of sentential arguments.

Consider the following argument.

(A) $P \rightarrow Q$
$Q \rightarrow R$
―――――
$P \rightarrow R$

This argument is valid, as can easily be demonstrated using truth-tables. Can we derive the conclusion from the premises? The following begins the derivation.

(1)	$P \rightarrow Q$	Pr
(2)	$Q \rightarrow R$	Pr
(3)	SHOW: $P \rightarrow R$	???
(4)	???	???

What formulas can we write down at line (4)? There are numerous formulas that follow from the premises according to the inference rules. But, not a single one of them makes any progress toward showing the conclusion P→R. In fact, upon close examination, we see that we have no means at our disposal to prove this argument. We are stuck.

In other words, as it currently stands, derivation system SL is inadequate. The above argument is valid, by truth-tables, but it cannot be proven in system SL. Accordingly, system SL must be strengthened so as to allow us to prove the above argument. Of course, we don't want to make the system so strong that we can derive invalid conclusions, so we have to be careful, as usual.

How might we argue for such a conclusion? Consider a concrete instance of the argument form.

(I) if the gas tank gets a hole, then the car runs out of gas;
if the car runs out of gas, then the car stops;
therefore, if the gas tank gets a hole, then the car stops.

In order to argue for the conclusion of (I), it seems natural to argue as follows. First, suppose the premises are true, in order to show the conclusion. The conclusion says that

the car stops **if** the gas tank gets a hole

or in other words,

the car stops **supposing** the gas tank gets a hole.

So, suppose also that the antecedent,

the gas tank gets a hole,

is true. In conjunction with the first premise, we can infer the following by *modus ponens* (→O):

the car runs out of gas.

And from this in conjunction with the second premise, we can infer the following by *modus ponens* (→O).

the car stops

So supposing the antecedent (the gas tank gets a hole), we have deduced the consequent (the car stops). In other words, we have shown the conclusion – if the gas tank gets a hole, then the car stops.

The above line of reasoning is made formal in the following official derivation.

Example 1

(1)	H → R	Pr
(2)	R → S	Pr
(3)	~~SHOW~~: H → S	CD
(4)	│ H	As
(5)	│ ~~SHOW~~: S	DD
(6)	│ │ R	1,4,→O
(7)	│ │ S	2,6,→O

This new-fangled derivation requires explaining. First of all, there are two show-lines; in particular, one derivation is nested inside another derivation. This is because the original problem – showing H→S – is reduced to another problem, showing S assuming H. This procedure is in accordance with a new show-rule, called conditional derivation, which may be intuitively formulated as follows.

> **Conditional Derivation** (Intuitive Formulation)
>
> In order to show a conditional $\mathcal{A}\to C$, it is sufficient to show the consequent C, assuming the antecedent $\mathcal{A}$.

The official formulation of conditional derivation is considerably more complicated, being given by the following two system rules.

System Rule 5 (a show-rule)

> **Conditional Derivation** (CD)
>
> If one has a show-line of the form 'SHOW: $\mathcal{A}\to C$', and one has C as a later available line, and there are no subsequent uncancelled show-lines, then one is entitled to box and cancel 'SHOW: $\mathcal{A}\to C$'.
> The annotation is 'CD'

System Rule 6 (an assumption rule)

> If one has a show-line of the form 'SHOW: $\mathcal{A}\to C$', then one is entitled to write down the antecedent $\mathcal{A}$ on the very next line, as an assumption.
> The annotation is 'As'

It is probably easier to understand conditional derivation by way of the associated picture.

Conditional Derivation (CD)

~~SHOW~~: $\mathcal{A} \to \mathcal{C}$	CD
$\mathcal{A}$	As
~~SHOW~~: $\mathcal{C}$	

This is supposed to depict the nature of conditional derivation; one shows a conditional $\mathcal{A}\to\mathcal{C}$ by assuming its antecedent $\mathcal{A}$ and showing its consequent $\mathcal{C}$.

In order to further our understanding of conditional derivation, we do a few examples.

Example 2

(1)	P → R	Pr
(2)	Q → S	Pr
(3)	~~SHOW~~: (P & Q) → (R & S)	CD
(4)	P & Q	As
(5)	~~SHOW~~: R & S	DD
(6)	P	4,&O
(7)	Q	4,&O
(8)	R	1,6,→O
(9)	S	2,7,→O
(10)	R & S	8,9,&I

Example 3

(1)	Q → R	Pr
(2)	R → (P → S)	Pr
(3)	~~SHOW~~: (P & Q) → S	CD
(4)	P & Q	As
(5)	~~SHOW~~: S	DD
(6)	P	4,&O
(7)	Q	4,&O
(8)	R	1,7,→O
(9)	P → S	2,8,→O
(10)	S	6,9,→O

The above examples involve two show-lines; each one involves a direct derivation inside a conditional derivation. The following examples introduce a new twist – three show-lines in the same derivation, with a conditional derivation inside a conditional derivation.

Example 4

(1)	(P & Q) → R	Pr
(2)	~~SHOW~~: P → (Q → R)	CD
(3)	│ P	As
(4)	│ ~~SHOW~~: Q → R	CD
(5)	│ │ Q	As
(6)	│ │ ~~SHOW~~: R	DD
(7)	│ │ │ P & Q	3,5,&I
(8)	│ │ │ R	1,7,→O

Example 5

(1)	(P & Q) → R	Pr
(2)	~~SHOW~~: (P → Q) → (P → R)	CD
(3)	│ P → Q	As
(4)	│ ~~SHOW~~: P → R	CD
(5)	│ │ P	As
(6)	│ │ ~~SHOW~~: R	DD
(7)	│ │ │ Q	3,5,→O
(8)	│ │ │ P & Q	5,7,&I
(9)	│ │ │ R	1,8,→O

Needless to say, the depth of nesting is not restricted; consider the following example.

Example 6

(1)	(P & Q) → (R → S)	Pr
(2)	~~SHOW~~: R → [(P → Q) → (P → S)]	CD
(3)	│ R	As
(4)	│ ~~SHOW~~: (P → Q) → (P → S)	CD
(5)	│ │ P → Q	As
(6)	│ │ ~~SHOW~~: P → S	CD
(7)	│ │ │ P	As
(8)	│ │ │ ~~SHOW~~: S	DD
(9)	│ │ │ │ Q	5,7,→O
(10)	│ │ │ │ P & Q	7,9,&I
(11)	│ │ │ │ R → S	1,10,→O
(12)	│ │ │ │ S	3,11,→O

Irrespective of the complexity of the above problems, they are solved in the same systematic manner. At each point where we come across 'SHOW: $\mathcal{A}$→$\mathcal{C}$', we immediately write down two more lines – we assume the antecedent, $\mathcal{A}$, in order to (attempt to) show the consequent, $\mathcal{C}$.

That is all there is to it!

12. INDIRECT DERIVATION (FIRST FORM)

System SL is now a *complete* set of rules for sentential logic; every valid argument of sentential logic can be proved valid in system SL. System SL is also *consistent*, which is to say that no invalid argument can be proven in system SL. Demonstrating these two very important logical facts – that system SL is both complete and consistent – is well outside the scope of introductory logic. It rather falls under the scope of *metalogic*, which is studied in more advanced courses in logic.

Even though system SL is complete as it stands, we will nonetheless enhance it further, thereby sacrificing elegance in favor of convenience. Consider the following argument form.

(a1) P → Q
P → ~Q
————
~P

Using truth-tables, one can quickly demonstrate that (a1) is valid. What happens when we try to construct a derivation that proves it to be valid? Consider the following start.

(1)	P → Q	Pr
(2)	P → ~Q	Pr
(3)	SHOW: ~P	???
(4)	???	???

An *attempted* derivation, using DD and CD, might go as follows.

> Consider line (3), which is a negation. We cannot show it by conditional derivation; it's not a conditional! That leaves direct derivation. Well, the premises are both conditionals, so the appropriate rule is arrow-out. But arrow-out requires a minor premise. In the case of (1) we need P or ~Q; in the case of (2), we need P or ~~Q; none of these is available. We are stuck!

We are trying to show ~P, which says in effect that P is false. Let's try a sneaky approach to the problem. Just for the helluvit, let us *assume* the opposite of what we are trying to show, and see what happens. So right below 'SHOW: ~P', we write P as an assumption. That yields the following partial derivation.

(1)	P → Q	Pr
(2)	P → ~Q	Pr
(3)	SHOW: ~P	???
(4)	P	As??
(6)	Q	1,4,→O
(7)	~Q	1,5,→O
(8)	Q & ~Q	5,6,&I

We have gotten down to line (8) which is Q&~Q. From our study of truth-tables, we know that this formula is a self-contradiction; it is false no matter what. So we see that assuming P at line (4) leads to a very bizarre result, a self-contradiction at line (8).

So, we have shown, in effect, that if P is true, then so is Q&~Q, which means that we have shown P→(Q&~Q). To see this, let us rewrite the problem as follows. Notice especially the new show-line (4).

(1)	P → Q	Pr
(2)	P → ~Q	Pr
(3)	SHOW: ~P	???
(4)	~~SHOW~~: P → (Q & ~Q)	CD
(5)	│ P	As
(6)	│ ~~SHOW~~: Q & ~Q	DD
(7)	│ │ Q	1,5,→O
(8)	│ │ ~Q	2,5,→O
(9)	│ │ Q & ~Q	7,8,&I

This is OK as far as it goes, but it is still not complete; show-line (3) has not been cancelled yet, which is marked in the annotation column by '???'. Line (4) is permitted, by the show-line rule (we can try to show anything!). Lines (5) and (6) then are written down in accordance with conditional derivation. The remaining lines are completely ordinary.

So how do we complete the derivation? We are trying to show ~P; we have in fact shown P→(Q&~Q); in other words, we have shown that if P is true, then so is Q&~Q. But the latter can't be true, so neither can the former (by *modus tollens*). This reasoning can be made formal in the following part derivation.

(1)	P → Q	Pr
(2)	P → ~Q	Pr
(3)	SHOW: ~P	DD
(4)	~~SHOW~~: P → (Q & ~Q)	CD
(5)	│ P	As
(6)	│ ~~SHOW~~: Q & ~Q	DD
(7)	│ │ Q	1,5,→O
(8)	│ │ ~Q	2,5,→O
(9)	│ │ Q & ~Q	7,8,&I
(10)	~(Q & ~Q)	???
(11)	~P	4,10,→O

This is an OK derivation, except for line (10), which has no justification. At this stage in the elaboration of system SL, we could introduce a new system rule that allows one to write ~($\mathcal{A}$&~$\mathcal{A}$) at any point in a derivation. This rule would work perfectly well, but it is not nearly as tidy as what we do instead. We choose instead to abbreviate the above chain of reasoning considerably, by introducing a further show-rule, called indirect derivation, whose intuitive formulation is given as follows.

> **Indirect Derivation** (First Form)
>
> Intuitive Formulation
>
> In order to show a negation $\sim\mathcal{A}$, it is sufficient to show *any contradiction*, assuming the un-negated formula, $\mathcal{A}$.

We must still provide the official formulation of indirect derivation, which as usual is considerably more complex; see below.

Recall that a contradiction is any formula whose truth table yields all F's in the output column. There are infinitely many contradictions in sentential logic. For this reason, at this point, it is convenient to introduce a new symbol into the vocabulary of sentential logic. In addition to the usual symbols – the letters, the connective symbols, and the parentheses – we introduce the symbol '※', in accordance with the following syntactic and semantic rules.

> Syntactic Rule: ※ is a formula.

> Semantic Rule: ※ is false no matter what.

[Alternatively, ※ is a "zero-place" logical connective, whose truth table always produces F.] In other words, ※ is a *generic contradiction*; it is equivalent to every contradiction.

With our new generic contradiction, we can reformulate Indirect Derivation as follows.

> **Indirect Derivation** (First Form)
>
> Second Formulation
>
> In order to show a negation $\sim\mathcal{A}$, it is sufficient to show ※, assuming the un-negated formula, $\mathcal{A}$.

In addition to the syntactic and semantic rules governing ※, we also need inference rules; in particular, as with the other logical symbols, we need an elimination rule, and an introduction rule. These are given as follows.

Contradiction-In (※I)

$\mathcal{A}$
$\sim\mathcal{A}$
―――
※

Contradiction-Out (※O)

※
――
$\mathcal{A}$

We will have little use for the elimination rule, ※O; it is included simply for symmetry. By contrast, the introduction rule, ※I, will be used extensively.

We are now in a position to write down the official formulation of indirect derivation of the first form (we discuss the second form in the next section).

System Rule 7 (a show rule)

Indirect Derivation (First Form)

If one has a show-line of the form '`SHOW`: $\sim\mathcal{A}$', then if one has ※ as a later available line, and there are no subsequent uncancelled show-lines, then one is entitled to cancel '`SHOW`: $\sim\mathcal{A}$' and box off all subsequent lines. The annotation is 'ID'.

System Rule 8 (an assumption rule)

If one has a show-line of the form '`SHOW`: $\sim\mathcal{A}$', then one is entitled to write down the un-negated formula $\mathcal{A}$ on the *very next* line, as an assumption. The annotation is 'As'.

As with earlier rules, we offer a pictorial abbreviation of indirect derivation as follows.

Indirect Derivation (First Form)

~~SHOW~~: $\sim\mathcal{A}$	ID
│ $\mathcal{A}$	As
│ ~~SHOW~~: ※	
│ │	

With our new rules in hand, let us now go back and do our earlier derivation in accordance with the new rules.

Example 1

(1)	$P \to Q$	Pr
(2)	$P \to \sim Q$	Pr
(3)	~~SHOW~~: $\sim P$	ID
(4)	│ P	As
(5)	│ ~~SHOW~~: ※	DD
(6)	│ │ Q	1,4,→O
(7)	│ │ $\sim Q$	2,4,→O
(8)	│ │ ※	6,7,※I

On line (3), we are trying to show ~P, which is a negation, so we do it by ID This entails writing down P on the next line as an assumption, and writing down 'SHOW: ※' on the following line. On line (8), we obtain ※ from lines (6) and (7), applying our new rule ※I.

Let's do another simple example.

Example 2

(1)	$P \to Q$	Pr
(2)	$Q \to \sim P$	Pr
(3)	~~SHOW~~: $\sim P$	ID
(4)	│ P	As
(5)	│ ~~SHOW~~: ※	DD
(6)	│ │ Q	1,4,→O
(7)	│ │ $\sim P$	2,6,→O
(8)	│ │ ※	4,7,※I

In the previous two examples, ※ is obtained from an atomic formula and its negation. Sometimes, ※ comes from more complex formulas, as in the following examples.

Example 3

(1)	~(P ∨ Q)	Pr
(2)	~~SHOW~~: ~P	ID
(3)	│ P	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ P ∨ Q	3,∨I
(6)	│ │ ※	1,5,※I

Here, ※ comes by ※I from P∨Q and ~(P∨Q).

Example 4

(1)	~(P & Q)	Pr
(2)	~~SHOW~~: P → ~Q	CD
(3)	│ P	As
(4)	│ ~~SHOW~~: ~Q	ID
(5)	│ │ Q	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ │ P & Q	3,5,&I
(8)	│ │ │ ※	1,7,※I

Here, ※ comes, by ※I, from P&Q and ~(P&Q).

13. INDIRECT DERIVATION (SECOND FORM)

In addition to indirect derivation of the first form, we also add indirect derivation of the second form, which is very similar to the first form. Consider the following derivation problem.

(1)	P → Q	Pr
(2)	~P → Q	Pr
(3)	SHOW: Q	???

The same problem as before arises; we have no simple means of dealing with either premise. (3) is atomic, so we must show it by direct derivation, but that approach comes to a screeching halt!

Once again, let's do something sneaky (but completely legal!), and see where that leads.

(1)	P → Q	Pr
(2)	~P → Q	Pr
(3)	SHOW: Q	???
(4)	SHOW: ~~Q	???

We have written down an additional show-line (which is completely legal, remember). The new problem facing us – to show ~~Q – appears much more promising; specifically, we are trying to show a negation, so we can attack it using indirect derivation, which yields the following part-derivation.

(1)	P → Q	Pr
(2)	~P → Q	Pr
(3)	SHOW: Q	???
(4)	~~SHOW~~: ~~Q	ID
(5)	│ ~Q	As
(6)	│ ~~SHOW~~: ※	DD
(7)	│ │ ~P	1,5,→O
(8)	│ │ ~~P	2,5,→O
(9)	│ │ ※	7,8,※I

The derivation is not complete. Line (3) is not cancelled. We are trying to show Q; we have in fact shown ~~Q. This is a near-hit because we can apply Double Negation to line (4) to get Q. This yields the following completed derivation.

(1)	P → Q	Pr
(2)	~P → Q	Pr
(3)	~~SHOW~~: Q	DD
(4)	│ ~~SHOW~~: ~~Q	ID
(5)	│ │ ~Q	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ │ ~P	1,5,→O
(8)	│ │ │ ~~P	2,5,→O
(9)	│ │ │ ※	7,8,※I
(10)	│ Q	4,DN

This derivation presents something completely novel. Upon getting to line (9), we have shown ~~Q, which is marked by cancelling the 'SHOW' and boxing off the associated derivation. We can now use the formula ~~Q in connection with the usual rules of inference. In this particular case, we apply double negation to obtain line (10). This is in accordance with the following principle.

> As soon as one cancels a show-line 'SHOW: $\mathcal{A}$', thus obtaining '~~SHOW~~: $\mathcal{A}$', the formula $\mathcal{A}$ is available, at least until the show-line itself gets boxed off.

In order to abbreviate the above derivation somewhat, we enhance the method of indirect derivation so as to include, in effect, the above double negation maneuver. The intuitive formulation of this rule is given as follows.

> **Indirect Derivation (Second Form)**
>
> Intuitive Formulation
>
> In order to show a formula $\mathcal{A}$, it is sufficient to show ※, assuming its negation ~$\mathcal{A}$.

As usual, the official formulation of the rule is more complex.

System Rule 9 (a show rule)

> **Indirect Derivation (Second Form)**
>
> If one has a show-line ‘SHOW: $\mathcal{A}$’, then if one has ※ as a later available line, and there are no intervening uncancelled show lines, then one is entitled to cancel ‘SHOW: $\mathcal{A}$’ and box off all subsequent formulas. The annotation is ‘ID’

System Rule 10 (an assumption rule)

> If one has a show-line ‘SHOW: $\mathcal{A}$’, then one is entitled to write down the negation $\sim\mathcal{A}$ on the *very next* line, as an assumption. The annotation is ‘As’

As usual, we also offer a pictorial version of the rule.

> **Indirect Derivation (Second Form)**
>
> ~~SHOW~~: $\mathcal{A}$
> | $\sim\mathcal{A}$
> | ~~SHOW~~: ※
> | |
> | |

With this new show-rule in hand, we can now rewrite our earlier derivation, as follows.

Example 1

(1)	P → Q	Pr
(2)	~P → Q	Pr
(3)	~~SHOW~~: Q	DD
(4)	| ~Q	As
(5)	| ~~SHOW~~: ※	DD
(6)	| | ~P	1,4,→O
(7)	| | ~~P	2,4,→O
(8)	| | ※	6,7,※I

In this particular problem, ※ is obtained by ※I from ~P and ~~P.

Let's look at one more example of the second form of indirect derivation.

Example 2

(1)	~(P & ~Q)	Pr
(2)	~~SHOW~~: P → Q	CD
(3)	│ P	As
(4)	│ ~~SHOW~~: Q	ID
(5)	│ │ ~Q	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ │ P & ~Q	3,5,&I
(8)	│ │ │ ※	1,7,※I

In this derivation we show P→Q by conditional derivation, which means we assume P and show Q. This is shown, in turn, by indirect derivation (second form), which means we assume ~Q to show ※. In this particular problem, ※ is obtained by ※I from P&~Q and ~(P&~Q).

14. SHOWING DISJUNCTIONS USING INDIRECT DERIVATION

The second form of ID is very useful for showing atomic formulas, as demonstrated in the previous section. It is also useful for showing disjunctions. Consider the following derivation problem.

(1)	~P → Q	Pr
(2)	SHOW: P ∨ Q	???

We are asked to show a disjunction P∨Q. CD is not available because this formula is not a conditional. ID of the first form is not available because it is not a negation. DD is available but it does not work (except in conjunction with the double-negation maneuver). That leaves the second form of ID, which yields the following.

(1)	~P → Q	Pr
(2)	SHOW: P ∨ Q	ID
(3)	~(P ∨ Q)	As
(4)	SHOW: ※	DD
(5)	???	

At this point, we are nearly stuck. We don't have the minor premise to deal with line (1), and we have no rule for dealing with line (3). So, what do we do? We can always write down a show-line of our own choosing, so we choose to write down 'SHOW: ~P'. This produces the following part-derivation.

(1)	~P → Q	Pr
(2)	SHOW: P ∨ Q	ID
(3)	~(P ∨ Q)	As
(4)	SHOW: ⋇	DD
(5)	~~SHOW~~: ~P	ID
(6)	| P	As
(7)	| ~~SHOW~~: ⋇	DD
(8)	| | P ∨ Q	6,∨I
(9)	| | ⋇	3,8,⋇I
(10)	???	

We are still not finished, but now we have shown ~P, so we can use it (while it is still available). This enables us to complete the derivation as follows.

(1)	~P → Q	Pr
(2)	~~SHOW~~: P ∨ Q	ID
(3)	| ~(P ∨ Q)	As
(4)	| ~~SHOW~~: ⋇	DD
(5)	| | ~~SHOW~~: ~P	ID
(6)	| | | P	As
(7)	| | | ~~SHOW~~: ⋇	DD
(8)	| | | | P ∨ Q	6,∨I
(9)	| | | | ⋇	3,8,⋇I
(10)	| | Q	1,5,→O
(11)	| | P ∨ Q	10,∨I
(12)	| | ⋇	3,11,⋇I

Lines 5-9 constitute a crucial, but completely routine, sub-derivation. Given how important, and yet how routine, this sub-derivation is, we now add a further inference-rule to our list. System SL is already complete as it stands, so we don't require this new rule. Adding it to system SL decreases its elegance. We add it purely for the sake of convenience.

The new rule is called tilde-wedge-out (~∨O). As its name suggests, it is a rule for breaking down formulas that are negations of disjunctions. It is pictorially presented as follows.

Tilde-Wedge-Out (~∨O)

$$\frac{\sim(\mathcal{A} \vee \mathcal{B})}{\sim\mathcal{A}} \qquad\qquad \frac{\sim(\mathcal{A} \vee \mathcal{B})}{\sim\mathcal{B}}$$

As with all inference rules, this rule applies exclusively to lines, not to parts of lines. In other words, the official formulation of the rule goes as follows.

> **Tilde-Wedge-Out** (~∨O)
>
> If one has available a line of the form ~($\mathcal{A}$ ∨ $\mathcal{B}$), then one is entitled to write down both ~$\mathcal{A}$ and ~$\mathcal{B}$.

Once we have the new rule ~∨O, the above derivation is much, much simpler.

Example 1

(1)	~P → Q	Pr
(2)	~~SHOW~~: P ∨ Q	ID
(3)	| ~(P ∨ Q)	As
(4)	| ~~SHOW~~: ※	DD
(5)	| | ~P	3,~∨O
(6)	| | ~Q	3,~∨O
(7)	| | Q	1,5,→O
(8)	| | ※	6,7,※I

In the above problem, we show a disjunction using the second form of indirect derivation. This involves a general strategy for showing any disjunction, formulated as follows.

General Strategy for Showing Disjunctions

> If you have a show-line of the form 'SHOW: $\mathcal{A}$∨$\mathcal{B}$', then use indirect derivation: first assume ~[$\mathcal{A}$∨$\mathcal{B}$], then write down 'SHOW: ※', then apply ~∨O to obtain ~$\mathcal{A}$ and ~$\mathcal{B}$, then proceed from there.

In cartoon form:

~~SHOW~~: $\mathcal{A}$ ∨ $\mathcal{B}$	ID
| ~[$\mathcal{A}$ ∨ $\mathcal{B}$]	As
| ~~SHOW~~: ※	
| | ~$\mathcal{A}$	~∨O
| | ~$\mathcal{B}$	~∨O
| |	

This particular strategy actually applies to any disjunction, simple or complex. In the previous example, the disjunction is simple (its disjuncts are atomic). In the next example, the disjunction is complex (its disjuncts are not atomic).

Example 2

(1)	(P ∨ Q) → (P & Q)	Pr
(2)	~~SHOW~~: (P & Q) ∨ (~P & ~Q)	ID
(3)	│ ~[(P & Q) ∨ (~P & ~Q)]	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ ~(P & Q)	3,~∨O
(6)	│ │ ~(~P & ~Q)	3,~∨O
(7)	│ │ ~(P ∨ Q)	1,5,→O
(8)	│ │ ~P	7,~∨O
(9)	│ │ ~Q	7,~∨O
(10)	│ │ ~P & ~Q	8,9,&I
(11)	│ │ ※	6,10,※I

The basic strategy is exactly like the previous problem. The only difference is that the formulas are more complex.

15. FURTHER RULES

In the previous section, we added the rule ~∨O to our list of inference rules. Although it is not strictly required, it does make a number of derivations much easier. In the present section, for the sake of symmetry, we add corresponding rules for the remaining two-place connectives; specifically, we add ~&O, ~→O, and ~↔O. That way, we have a rule for handling any negated molecular formula.

Also, we add one more rule that is sometimes useful, the Rule of Repetition.

The additional negation rules are given as follows.

Tilde-Ampersand-Out (~&O)

$$\frac{\sim(\mathcal{A} \;\&\; \mathcal{B})}{\mathcal{A} \rightarrow \sim\mathcal{B}}$$

Tilde-Arrow-Out (~→O)

$$\frac{\sim(\mathcal{A} \rightarrow \mathcal{C})}{\mathcal{A} \;\&\; \sim\mathcal{C}}$$

Tilde-Double-Arrow-Out (~↔O)

$$\frac{\sim(\mathcal{A} \leftrightarrow \mathcal{B})}{\sim\mathcal{A} \leftrightarrow \mathcal{B}}$$

The reader is urged to verify that these are all valid argument forms of sentential logic. There are other valid forms that could serve equally well as the rules in question. The choice is to a certain arbitrary. The advantage of the particular choice becomes more apparent in a later chapter on predicate logic.

Finally in this section, we officially present the Rule of Repetition.

Repetition (R)

$$\frac{\mathcal{A}}{\mathcal{A}}$$

In other words, if you have an available formula, $\mathcal{A}$, you can simply copy (repeat) it at any later time. See Problem #120 for an application of this rule.

16. SHOWING CONJUNCTIONS AND BICONDITIONALS

In the previous sections, strategies are suggested for showing various kinds of formulas, as follows.

Formula Type	Strategy
Conditional	Conditional Derivation
Negation	Indirect Derivation (1)
Atomic Formula	Indirect Derivation (2)
Disjunction	Indirect Derivation (2)

That leaves only two kinds of formulas – conjunctions and biconditionals. In the present section, we discuss the strategies for these kinds of formulas.

Strategy for Showing Conjunctions

If you have a show-line of the form 'SHOW: $\mathcal{A}$&$\mathcal{B}$', then write down two further show-lines. Specifically, first write down 'SHOW: $\mathcal{A}$' and complete the associated derivation, then write down 'SHOW: $\mathcal{B}$' and complete the associated derivation. Finally, apply &I, and cancel 'SHOW: $\mathcal{A}$&$\mathcal{B}$' by direct derivation.

This strategy is easier to see in its cartoon version.

~~SHOW~~: $\mathcal{A}$ & $\mathcal{B}$ DD
| ~~SHOW~~: $\mathcal{A}$
| |
| ~~SHOW~~: $\mathcal{B}$
| |
| $\mathcal{A}$ & $\mathcal{B}$ &I

There is a parallel strategy for biconditionals, given as follows.

Strategy for Showing Biconditionals

If you have a show-line of the form 'SHOW: $\mathcal{A}\leftrightarrow\mathcal{B}$', then write down two further show-lines. Specifically, first write down 'SHOW: $\mathcal{A}\rightarrow\mathcal{B}$' and complete the associated derivation, then write down 'SHOW: $\mathcal{B}\rightarrow\mathcal{A}$' and complete the associated derivation. Finally, apply ↔I and cancel 'SHOW: $\mathcal{A}\leftrightarrow\mathcal{B}$' by direct derivation.

The associated cartoon version is as follows.

~~SHOW~~: $\mathcal{A} \leftrightarrow \mathcal{B}$ DD
| ~~SHOW~~: $\mathcal{A} \rightarrow \mathcal{B}$
| |
| ~~SHOW~~: $\mathcal{B} \rightarrow \mathcal{A}$
| |
| $\mathcal{A} \leftrightarrow \mathcal{B}$ ↔I

We conclude this section by doing a few examples that use these two strategies.

Example 1

(1)	(A ∨ B) → C	Pr
(2)	~~SHOW~~: (A → C) & (B → C)	DD
(3)	│ ~~SHOW~~: A → C	CD
(4)	│ │ A	As
(5)	│ │ ~~SHOW~~: C	DD
(6)	│ │ │ A ∨ B	4,∨I
(7)	│ │ │ C	1,6,→O
(8)	│ ~~SHOW~~: B → C	CD
(9)	│ │ B	As
(10)	│ │ ~~SHOW~~: C	DD
(11)	│ │ │ A ∨ B	9,∨I
(12)	│ │ │ C	1,11,→O
(13)	│ (A → C) & (B → C)	3,8,&I

Example 2

(1)	~P → Q	Pr
(2)	Q → ~P	Pr
(3)	~~SHOW~~: P ↔ ~Q	DD
(4)	│ ~~SHOW~~: P → ~Q	CD
(5)	│ │ P	As
(6)	│ │ ~~SHOW~~: ~Q	DD
(7)	│ │ │ ~~P	5,DN
(8)	│ │ │ ~Q	2,7,→O
(9)	│ ~~SHOW~~: ~Q → P	CD
(10)	│ │ ~Q	As
(11)	│ │ ~~SHOW~~: P	DD
(12)	│ │ │ ~~P	1,10,→O
(13)	│ │ │ P	12,DN
(14)	│ P ↔ ~Q	4,9,↔I

Example 3

(1)	$(P \& Q) \rightarrow \sim R$	Pr
(2)	$Q \rightarrow R$	Pr
(3)	~~SHOW~~: $P \leftrightarrow (P \& \sim Q)$	DD
(4)	~~SHOW~~: $P \rightarrow (P \& \sim Q)$	CD
(5)	P	As
(6)	~~SHOW~~: $P \& \sim Q$	DD
(7)	~~SHOW~~: $\sim Q$	ID
(8)	Q	As
(9)	~~SHOW~~: ※	DD
(10)	$P \& Q$	5,8,&I
(11)	$\sim R$	1,10,→O
(12)	R	2,8,→O
(13)	※	11,12,※I
(14)	$P \& \sim Q$	5,7,&I
(15)	~~SHOW~~: $(P \& \sim Q) \rightarrow P$	CD
(16)	$P \& \sim Q$	As
(17)	~~SHOW~~: P	DD
(18)	P	16,&O
(19)	$P \leftrightarrow (P \& \sim Q)$	4,15,↔I

17. THE WEDGE-OUT STRATEGY

We now have a strategy for dealing with every kind of show-line, whether it be atomic, a negation, a conjunction, a disjunction, a conditional, or a biconditional.

One often runs into problems that do not immediately surrender to any of these strategies. Consider the following problem, partly completed.

(1)	$(P \rightarrow Q) \vee (P \rightarrow R)$	Pr
(2)	SHOW: $(P \& \sim Q) \rightarrow R$	CD
(3)	$P \& \sim Q$	As
(4)	SHOW: R	ID
(5)	$\sim R$	As
(6)	SHOW: ※	DD
(7)	P	3,&O
(8)	$\sim Q$	3,&O
(9)	???	???

Everything goes smoothly until we reach line (9), at which point we are stuck. The premise is a disjunction; so in order to decompose it by wedge-out, we need one of the minor premises; that is, we need either $\sim(P \rightarrow Q)$ or $\sim(P \rightarrow R)$. If we had, say, the first one, then we could proceed as follows.

(1)	(P → Q) ∨ (P → R)	Pr
(2)	SHOW: (P & ~Q) → R	CD
(3)	P & ~Q	As
(4)	SHOW: R	ID
(5)	~R	As
(6)	SHOW: ※	DD
(7)	P	3,&O
(8)	~Q	3,&O
(9)	~(P → Q)	?????
(10)	P → R	1,9,∨O
(11)	R	7,10,→O
(12)	※	5,11,※I

This is great, except for line (9), which is completely without justification! For this reason the derivation remains incomplete. However, if we could somehow get ~(P→Q), then the derivation could be legally completed. So what can we do? One thing is to try to *show* the needed formula. Remember, one can write down any show-line whatsoever. Doing this produces the following partly completed derivation.

(1)	(P → Q) ∨ (P → R)	Pr
(2)	SHOW (P & ~Q) → R	CD
(3)	P & ~Q	As
(4)	SHOW: R	ID
(5)	~R	As
(6)	SHOW: ※	DD
(7)	P	3,&O
(8)	~Q	3,&O
(9)	~~SHOW~~: ~(P → Q)	ID
(10)	| P → Q	As
(11)	| ~~SHOW~~: ※	DD
(12)	| | Q	7,10,→O
(13)	| | ※	8,12,※I

Notice that we have shown exactly what we needed, so we can use it to complete the derivation as follows.

Example 1

(1)	(P → Q) ∨ (P → R)	Pr
(2)	~~SHOW~~: (P & ~Q) → R	CD
(3)	│ P & ~Q	As
(4)	│ ~~SHOW~~: R	ID
(5)	│ │ ~R	As
(6)	│ │ ~~SHOW~~: ✳	DD
(7)	│ │ │ P	3,&O
(8)	│ │ │ ~Q	3,&O
(9)	│ │ │ ~~SHOW~~: ~(P → Q)	ID
(10)	│ │ │ │ P → Q	As
(11)	│ │ │ │ ~~SHOW~~: ✳	DD
(12)	│ │ │ │ │ Q	7,10,→O
(13)	│ │ │ │ │ ✳	8,12,✳I
(14)	│ │ │ P → R	1,9,∨O
(15)	│ │ │ ~P	5,14,→O
(16)	│ │ │ ✳	7,15,✳I

The above derivation is an example of a general strategy, called the wedge-out strategy, which is formulated as follows.

> **Wedge-Out Strategy**
>
> If you have as an available line a disjunction 𝒜∨ℬ, then look for means to break it down using wedge-out. This requires having either ~𝒜 or ~ℬ. Look for ways to get one of these. If you get stuck, try to *show* one of them; i.e., write 'SHOW: ~𝒜' or 'SHOW: ~ℬ'.

In pictures, this strategy looks thus:

𝒜 ∨ ℬ	𝒜 ∨ ℬ
SHOW: 𝒞	SHOW: 𝒞
o	o
o	o
~~SHOW~~: ~𝒜	~~SHOW~~: ~ℬ
│	│
ℬ ∨O	𝒜 ∨O
o	o
o	o
o	o

How does one decide which one to show; the rule of thumb (not absolutely reliable, however) is this:

> **Rule of Thumb**
>
> In the wedge-out strategy, the choice of which disjunct to attack is largely unimportant, so you might as well choose the first one.

Since the wedge-out strategy is so important, let's do one more example. Here the crucial line is line (7).

Example 2

(1)	(P & R) ∨ (Q & R)	Pr
(2)	~~SHOW~~: ~P → Q	CD
(3)	│ ~P	As
(4)	│ ~~SHOW~~: Q	ID
(5)	│ │ ~Q	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ │ ~~SHOW~~: ~(P & R)	ID
(8)	│ │ │ │ P & R	As
(9)	│ │ │ │ ~~SHOW~~: ※	DD
(10)	│ │ │ │ │ P	8,&O
(11)	│ │ │ │ │ ※	3,10,※I
(12)	│ │ │ │ Q & R	1,7,∨O
(13)	│ │ │ │ Q	12,&O
(14)	│ │ │ │ ※	5,13,※I

18. THE ARROW-OUT STRATEGY

There is one more strategy that we will examine, one that is very similar to the wedge-out strategy; the difference is that it pertains to conditionals.

> **Arrow-Out Strategy**
>
> If you have as an available line a conditional $\mathcal{A}\rightarrow\mathcal{C}$, then look for means to break it down using arrow-out. This requires having either $\mathcal{A}$ or $\sim\mathcal{C}$. Look for ways to get one of these. If you get stuck, try to *show* one of them; i.e., write 'SHOW: $\mathcal{A}$' or 'SHOW: $\sim\mathcal{C}$'.

In pictures:

$\mathcal{A} \rightarrow \mathcal{C}$		$\mathcal{A} \rightarrow \mathcal{C}$	
SHOW: $\mathcal{B}$		SHOW: $\mathcal{B}$	
o		o	
o		o	
~~SHOW~~: $\mathcal{A}$		~~SHOW~~: $\sim\mathcal{C}$	
$\mathcal{C}$	→O	$\sim\mathcal{A}$	→O
o		o	
o		o	
o		o	

The following is a derivation that employs the arrow-out strategy. The crucial line is line (5).

Example 1

(1)	(P → Q) → (P → R)	Pr
(2)	~~SHOW~~: (P & Q) → R	CD
(3)	| P & Q	As
(4)	| ~~SHOW~~: R	DD
(5)	| | ~~SHOW~~: P → Q	CD
(6)	| | | P	As
(7)	| | | ~~SHOW~~: Q	DD
(8)	| | | | Q	3,&O
(9)	| | P → R	1,5,→O
(10)	| | P	3,&O
(11)	| | R	9,10,→O

19. SUMMARY OF THE SYSTEM RULES FOR SYSTEM SL

1. System Rule 1 (The Premise Rule)

At any point in a derivation, ***prior*** to the first show-line, any premise may be written down. The annotation is 'Pr'.

2. System Rule 2 (The Inference-Rule Rule)

At any point in a derivation, a formula may be written down if it follows from previous available lines by an inference rule. The annotation cites the lines numbers, and the inference rule, in that order.

3. System Rule 3 (The Show-Line Rule)

At *any* point in a derivation, one is entitled to write down the expression 'SHOW: $\mathcal{A}$',
for *any* formula $\mathcal{A}$ *whatsoever*.

4. System Rule 4 (a show-rule)

Direct Derivation (DD)

If one has a show-line 'SHOW: $\mathcal{A}$', and one obtains $\mathcal{A}$ as a later available line, and there are no intervening uncancelled show-lines, then one is entitled to box and cancel 'SHOW: $\mathcal{A}$'. The annotation is 'DD'

5. System Rule 5 (a show-rule)

Conditional Derivation (CD)

If one has a show-line of the form 'SHOW: $\mathcal{A}\rightarrow\mathcal{C}$', and one has $\mathcal{C}$ as a later available line, and there are no subsequent uncancelled show-lines, then one is entitled to box and cancel 'SHOW: $\mathcal{A}\rightarrow\mathcal{C}$'. The annotation is 'CD'

6. System Rule 6 (an assumption rule)

If one has a show-line of the form 'SHOW: $\mathcal{A}\rightarrow\mathcal{C}$', then one is entitled to write down the antecedent $\mathcal{A}$ on the very next line, as an assumption. The annotation is 'As'

7. System Rule 7 (a show rule)

Indirect Derivation (First Form)

If one has a show-line of the form 'SHOW: $\sim\mathcal{A}$', then if one has ※ as a later available line, and there are no intervening uncancelled show-lines, then one is entitled to box and cancel 'SHOW: $\sim\mathcal{A}$'. The annotation is 'ID'.

8. System Rule 8 (an assumption rule)

If one has a show-line of the form 'SHOW: $\sim\mathcal{A}$', then one is entitled to write down the un-negated formula $\mathcal{A}$ on the *very next* line, as an assumption. The annotation is 'As'

9. System Rule 9 (a show rule)

Indirect Derivation (Second Form)

If one has a show-line 'SHOW: $\mathcal{A}$', then if one has ※ as a later available line, and there are no intervening uncancelled show lines, then one is entitled to box and cancel 'SHOW: $\mathcal{A}$'. The annotation is 'ID'

10. System Rule 10 (an assumption rule)

If one has a show-line 'SHOW: $\mathcal{A}$', then one is entitled to write down the negation $\sim\mathcal{A}$ on the *very next* line, as an assumption. The annotation is 'As'

11. System Rule 11 (Definition of available formula)

Formula $\mathcal{A}$ in a derivation is ***available*** if and only if either $\mathcal{A}$ occurs (as a whole line!), but is not inside a box, or '~~SHOW~~: $\mathcal{A}$' occurs (as a whole line!), but is not inside a box.

12. System Rule 12 (definition of box-and-cancel)

To ***box and cancel*** a show-line 'SHOW: $\mathcal{A}$' is to strike through 'SHOW' resulting in '~~SHOW~~', and box off *all* lines below 'SHOW: $\mathcal{A}$' (which is to say all lines *at the time* the box-and-cancel occurs).

20. PICTORIAL SUMMARY OF THE RULES OF SYSTEM SL

INITIAL INFERENCE RULES

Ampersand-In (&I)

$$\frac{\mathcal{A} \quad \mathcal{B}}{\mathcal{A} \,\&\, \mathcal{B}} \qquad \frac{\mathcal{A} \quad \mathcal{B}}{\mathcal{B} \,\&\, \mathcal{A}}$$

Ampersand-Out (&O)

$$\frac{\mathcal{A} \,\&\, \mathcal{B}}{\mathcal{A}} \qquad \frac{\mathcal{A} \,\&\, \mathcal{B}}{\mathcal{B}}$$

Wedge-In (∨I)

$$\frac{\mathcal{A}}{\mathcal{A} \vee \mathcal{B}} \qquad \frac{\mathcal{A}}{\mathcal{B} \vee \mathcal{A}}$$

Wedge-Out (∨O)

$$\frac{\mathcal{A} \vee \mathcal{B} \quad \sim\mathcal{A}}{\mathcal{B}} \qquad \frac{\mathcal{A} \vee \mathcal{B} \quad \sim\mathcal{B}}{\mathcal{A}}$$

Double-Arrow-In (↔I)

$$\frac{\mathcal{A} \rightarrow \mathcal{B} \quad \mathcal{B} \rightarrow \mathcal{A}}{\mathcal{A} \leftrightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \rightarrow \mathcal{B} \quad \mathcal{B} \rightarrow \mathcal{A}}{\mathcal{B} \leftrightarrow \mathcal{A}}$$

Double-Arrow-Out (↔O)

$$\frac{\mathcal{A} \leftrightarrow \mathcal{B}}{\mathcal{A} \rightarrow \mathcal{B}} \qquad \frac{\mathcal{A} \leftrightarrow \mathcal{B}}{\mathcal{B} \rightarrow \mathcal{A}}$$

Arrow-Out (→O)

$$\frac{\mathcal{A} \rightarrow \mathcal{C} \quad \mathcal{A}}{\mathcal{C}} \qquad \frac{\mathcal{A} \rightarrow \mathcal{C} \quad \sim\mathcal{C}}{\sim\mathcal{A}}$$

Double Negation (DN)

$$\frac{\mathcal{A}}{\sim\sim\mathcal{A}} \qquad \frac{\sim\sim\mathcal{A}}{\mathcal{A}}$$

ADDITIONAL INFERENCE RULES

Contradiction-In (※I)

$$\frac{\mathcal{A} \quad \sim\mathcal{A}}{※}$$

Contradiction-Out (※O)

$$\frac{※}{\mathcal{A}}$$

Tilde-Wedge-Out (~∨O)

$$\frac{\sim(\mathcal{A} \vee \mathcal{B})}{\sim\mathcal{A}} \qquad \frac{\sim(\mathcal{A} \vee \mathcal{B})}{\sim\mathcal{B}}$$

Tilde-Ampersand-Out (~&O)

$$\frac{\sim(\mathcal{A} \,\&\, \mathcal{B})}{\mathcal{A} \rightarrow \sim\mathcal{B}}$$

Tilde-Arrow-Out (~→O)

$$\frac{\sim(\mathcal{A} \rightarrow \mathcal{C})}{\mathcal{A} \,\&\, \sim\mathcal{C}}$$

Tilde-Double-Arrow-Out (~↔O)

$$\frac{\sim(\mathcal{A} \leftrightarrow \mathcal{B})}{\sim\mathcal{A} \leftrightarrow \mathcal{B}}$$

Repetition (R)

$$\frac{\mathcal{A}}{\mathcal{A}}$$

SHOW-RULES

Direct Derivation (DD)

~~SHOW~~: $\mathcal{A}$	DD
|	
| $\mathcal{A}$	

Conditional Derivation (CD)

~~SHOW~~: $\mathcal{A} \rightarrow \mathcal{C}$	CD
| $\mathcal{A}$	As
| ~~SHOW~~: $\mathcal{C}$	
| |	

Indirect Derivation (First Form)

~~SHOW~~: $\sim\mathcal{A}$	ID
| $\mathcal{A}$	As
| ~~SHOW~~: ※	
| |	

Indirect Derivation (Second Form)

~~SHOW~~: $\mathcal{A}$	ID
| $\sim\mathcal{A}$	As
| ~~SHOW~~: ※	
| |	

21. PICTORIAL SUMMARY OF STRATEGIES

~~SHOW~~: $\mathcal{A} \& \mathcal{B}$ DD
| ~~SHOW~~: $\mathcal{A}$
| |
| ~~SHOW~~: $\mathcal{B}$
| |
| $\mathcal{A} \& \mathcal{B}$ &I

~~SHOW~~: $\mathcal{A} \rightarrow \mathcal{C}$ CD
| A As
| ~~SHOW~~: $\mathcal{C}$
| |

~~SHOW~~: $\mathcal{A} \vee \mathcal{B}$ ID
| $\sim[\mathcal{A} \vee \mathcal{B}]$ As
| ~~SHOW~~: ⨳
| | $\sim\mathcal{A}$ $\sim\vee$O
| | $\sim\mathcal{B}$ $\sim\vee$O
| |

~~SHOW~~: $\mathcal{A} \leftrightarrow \mathcal{B}$ DD
| ~~SHOW~~: $\mathcal{A} \rightarrow \mathcal{B}$
| |
| ~~SHOW~~: $\mathcal{B} \rightarrow \mathcal{A}$
| |
| $\mathcal{A} \leftrightarrow \mathcal{B}$ $\leftrightarrow$I

~~SHOW~~: $\sim\mathcal{A}$	ID
$\mathcal{A}$	As
~~SHOW~~: ※	

~~SHOW~~: A	ID
$\sim$A	As
~~SHOW~~: ※	

Wedge-Out Strategy

Wedge-Out Strategy

If you have as an available line a disjunction $\mathcal{A}\vee\mathcal{B}$, then look for means to break it down using wedge-out. This requires having either $\sim\mathcal{A}$ or $\sim\mathcal{B}$. Look for ways to get one of these. If you get stuck, try to *show* one of them; i.e., write 'SHOW: $\sim\mathcal{A}$' or 'SHOW: $\sim\mathcal{B}$'.

$\mathcal{A}\vee\mathcal{B}$		$\mathcal{A}\vee\mathcal{B}$	
SHOW: $\mathcal{C}$		SHOW: $\mathcal{C}$	
o		o	
o		o	
~~SHOW~~: $\sim\mathcal{A}$		~~SHOW~~: $\sim\mathcal{B}$	
$\mathcal{B}$	$\vee$O	$\mathcal{A}$	$\vee$O
o		o	
o		o	
o		o	

Arrow-Out Strategy

> If you have as an available line a conditional $\mathcal{A}\to\mathcal{B}$, then look for means to break it down using arrow-out. This requires having either $\mathcal{A}$ or $\sim\mathcal{B}$. Look for ways to get one of these. If you get stuck, try to *show* one of them; i.e., write 'SHOW: $\mathcal{A}$' or 'SHOW: $\sim\mathcal{B}$'.

$\mathcal{A} \to \mathcal{C}$	$\mathcal{A} \to \mathcal{C}$
SHOW: $\mathcal{B}$	SHOW: $\mathcal{B}$
o	o
o	o
~~SHOW~~: $\mathcal{A}$	~~SHOW~~: $\sim\mathcal{C}$
|	|
$\mathcal{C}$ →O	$\sim\mathcal{A}$ →O
o	o
o	o
o	o

22. EXERCISES FOR CHAPTER 5

EXERCISE SET A (Simple Derivation)

For each of the following arguments, construct a simple derivation of the conclusion (marked by '/') from the premises, using the simple rules MP, MT, MTP1, and MTP2.

(1) P ; P → Q ; Q → R ; R → S / S

(2) P → Q ; Q → R ; R → S ; ~S / ~P

(3) ~P ∨ Q ; ~Q ; P ∨ R / R

(4) P ∨ Q ; ~P ; Q → R / R

(5) P ; P → ~Q ; R → Q ; ~R → S / S

(6) P ∨ ~Q ; ~P ; R → Q ; ~R → S / S

(7) (P → Q) → P ; P → Q / Q

(8) (P → Q) → R ; R → P ; P → Q / Q

(9) (P → Q) → (Q → R) ; P → Q ; P / R

(10) ~P → Q ; ~Q ; R ∨ ~P / R

(11) ~P → (~Q ∨ R) ; P → R ; ~R / ~Q

(12) P → ~Q ; ~S → P ; ~~Q / ~~S

(13) P ∨ Q ; Q → R ; ~R / P

(14) ~P → (Q ∨ R) ; P → Q ; ~Q / R

(15) P → R ; ~P → (S ∨ R) ; ~R / S

(16) P ∨ ~Q ; ~R → ~~Q ; R → ~S ; ~~S / P

(17) (P → Q) ∨ (R → S) ; (P → Q) → R ; ~R / R → S

(18) (P → Q) → (R → S) ; (R → T) ∨ (P → Q) ; ~(R → T) / R → S

(19) ~R → (P ∨ Q) ; R → P ; (R → P) → ~P / Q

(20) (P → Q) ∨ R ; [(P → Q) ∨ R] → ~R ; (P → Q) → (Q → R) / ~Q

EXERCISE SET B (Direct Derivation)

Convert each of the simple derivations in Exercise Set A into a direct derivation; use the introduction-elimination rules.

EXERCISE SET C (Direct Derivation)

Directions for remaining exercises: For each of the following arguments, construct a derivation of the conclusion (marked by '/') from the premises, using the rules of System SL.

(21) P & Q ; P → (R & S) / Q & S

(22) P & Q ; (P ∨ R) → S / P & S

(23) P ; (P ∨ Q) → (R & S) ; (R ∨ T) → U / U

(24) P → Q ; P ∨ R ; ~Q / R & ~P

(25) P → Q ; ~R → (Q → S) ; R → T ; ~T & P / Q & S

(26) P → Q ; R ∨ ~Q ; ~R & S ; (~P & S) → T / T

(27) P ∨ ~Q ; ~R → Q ; R → ~S ; S / P

(28) P & Q ; (P ∨ T) → R ; S → ~R / ~S

(29) P & Q ; P → R ; (P & R) → S / Q & S

(30) P → Q ; Q ∨ R ; (R & ~P) → S ; ~Q / S

(31) P & Q / Q & P

(32) P & (Q & R) / (P & Q) & R

(33) P / P & P

(34) P / P & (P ∨ Q)

(35) P & ~P / Q

(36) P ↔ ~Q ; Q ; P ↔ ~S / S

(37) P & ~Q ; Q ∨ (P → S) ; (R & T) ↔ S / P & R

(38) P → Q ; (P → Q) → (Q → P) ; (P ↔ Q) → P / P & Q

(39) ~P & Q ; (R ∨ Q) → (~S → P) ; ~S ↔ T / ~T

(40) P & ~Q ; Q ∨ (R → S) ; ~V → ~P ; V → (S → R) ; (R ↔ S) → T ;
U ↔ (~Q & T) / U

EXERCISE SET D (Conditional Derivation)

(41) $(P \vee Q) \rightarrow R$ / $Q \rightarrow R$

(42) $Q \rightarrow R$ / $(P \& Q) \rightarrow (P \& R)$

(43) $P \rightarrow Q$ / $(Q \rightarrow R) \rightarrow (P \rightarrow R)$

(44) $P \rightarrow Q$ / $(R \rightarrow P) \rightarrow (R \rightarrow Q)$

(45) $(P \& Q) \rightarrow R$ / $P \rightarrow (Q \rightarrow R)$

(46) $P \rightarrow (Q \rightarrow R)$ / $(P \rightarrow Q) \rightarrow (P \rightarrow R)$

(47) $(P \& Q) \rightarrow R$ / $[(P \rightarrow Q) \rightarrow P] \rightarrow [(P \rightarrow Q) \rightarrow R]$

(48) $(P \& Q) \rightarrow (R \rightarrow S)$ / $(P \rightarrow Q) \rightarrow [(P \& R) \rightarrow S]$

(49) $[(P \& Q) \& R] \rightarrow S$ / $P \rightarrow [Q \rightarrow (R \rightarrow S)]$

(50) $({\sim}P \& Q) \rightarrow R$ / $({\sim}Q \rightarrow P) \rightarrow ({\sim}P \rightarrow R)$

EXERCISE SET E (Indirect Derivation – First Form)

(51) $P \rightarrow Q$; $P \rightarrow {\sim}Q$ / ${\sim}P$

(52) $P \rightarrow Q$; $Q \rightarrow {\sim}P$ / ${\sim}P$

(53) $P \rightarrow Q$; ${\sim}Q \vee {\sim}R$; $P \rightarrow R$ / ${\sim}P$

(54) $P \rightarrow R$; $Q \rightarrow {\sim}R$ / ${\sim}(P \& Q)$

(55) $P \& Q$ / ${\sim}(P \rightarrow {\sim}Q)$

(56) $P \& {\sim}Q$ / ${\sim}(P \rightarrow Q)$

(57) ${\sim}P$ / ${\sim}(P \& Q)$

(58) ${\sim}P \& {\sim}Q$ / ${\sim}(P \vee Q)$

(59) $P \leftrightarrow Q$; ${\sim}Q$ / ${\sim}(P \vee Q)$

(60) $P \& Q$ / ${\sim}({\sim}P \vee {\sim}Q)$

(61) ${\sim}P \vee {\sim}Q$ / ${\sim}(P \& Q)$

(62) $P \vee Q$ / ${\sim}({\sim}P \& {\sim}Q)$

(63) $P \rightarrow Q$ / ${\sim}(P \& {\sim}Q)$

(64) $P \rightarrow (Q \rightarrow {\sim}P)$ / $P \rightarrow {\sim}Q$

(65) $(P \& Q) \rightarrow R$ / $(P \& {\sim}R) \rightarrow {\sim}Q$

(66) $(P \& Q) \rightarrow {\sim}R$ / $P \rightarrow {\sim}(Q \& R)$

(67) $P \rightarrow (Q \rightarrow R)$ / $(Q \& {\sim}R) \rightarrow {\sim}P$

(68) $P \rightarrow {\sim}(Q \& R)$ / $(P \& Q) \rightarrow {\sim}R$

(69) $P \rightarrow {\sim}(Q \& R)$ / $(P \rightarrow Q) \rightarrow (P \rightarrow {\sim}R)$

(70) $P \rightarrow (Q \rightarrow R)$ / $(P \rightarrow {\sim}R) \rightarrow (P \rightarrow {\sim}Q)$

EXERCISE SET F (Indirect Derivation – Second Form)

(71) $P \rightarrow Q$; $\sim P \rightarrow Q$ / Q

(72) $P \vee Q$; $P \rightarrow R$; $Q \vee \sim R$ / Q

(73) $\sim P \rightarrow R$; $Q \rightarrow R$; $P \rightarrow Q$ / R

(74) $(P \vee \sim Q) \rightarrow (R \,\&\, \sim S)$; $Q \vee S$ / Q

(75) $(P \vee Q) \rightarrow (R \rightarrow S)$; $(\sim S \vee T) \rightarrow (P \,\&\, R)$ / S

(76) $\sim(P \,\&\, \sim Q)$ / $P \rightarrow Q$

(77) $P \rightarrow (\sim Q \rightarrow R)$ / $(P \,\&\, \sim R) \rightarrow Q$

(78) $P \,\&\, (Q \vee R)$ / $\sim(P \,\&\, Q) \rightarrow R$

(79) $P \vee Q$ / $Q \vee P$

(80) $\sim P \rightarrow Q$ / $P \vee Q$

(81) $\sim(P \,\&\, Q)$ / $\sim P \vee \sim Q$

(82) $P \rightarrow Q$ / $\sim P \vee Q$

(83) $P \vee Q$; $P \rightarrow R$; $Q \rightarrow S$ / $R \vee S$

(84) $\sim P \rightarrow Q$; $P \rightarrow R$ / $Q \vee R$

(85) $\sim P \rightarrow Q$; $\sim R \rightarrow S$; $\sim Q \vee \sim S$ / $P \vee R$

(86) $(P \,\&\, \sim Q) \rightarrow R$ / $P \rightarrow (Q \vee R)$

(87) $\sim P \rightarrow (\sim Q \vee R)$ / $Q \rightarrow (P \vee R)$

(88) $P \,\&\, (Q \vee R)$ / $(P \,\&\, Q) \vee R$

(89) $(P \vee Q) \,\&\, (P \vee R)$ / $P \vee (Q \,\&\, R)$

(90) $(P \vee Q) \rightarrow (P \,\&\, Q)$ / $(P \,\&\, Q) \vee (\sim P \,\&\, \sim Q)$

EXERCISE SET G (Strategies)

(91) $P \rightarrow (Q \,\&\, R)$ / $(P \rightarrow Q) \,\&\, (P \rightarrow R)$

(92) $(P \vee Q) \rightarrow R$ / $(P \rightarrow R) \,\&\, (Q \rightarrow R)$

(93) $(P \vee Q) \rightarrow (P \,\&\, Q)$ / $P \leftrightarrow Q$

(94) $P \leftrightarrow Q$ / $Q \leftrightarrow P$

(95) $P \leftrightarrow Q$ / $\sim P \leftrightarrow \sim Q$

(96) $P \leftrightarrow Q$; $Q \rightarrow \sim P$ / $\sim P \,\&\, \sim Q$

(97) $(P \rightarrow Q) \vee (\sim Q \rightarrow R)$ / $P \rightarrow (Q \vee R)$

(98) $P \vee Q$; $P \rightarrow \sim Q$ / $(P \rightarrow Q) \rightarrow (Q \,\&\, \sim P)$

(99) $P \vee Q$; $\sim(P \,\&\, Q)$ / $(P \rightarrow Q) \rightarrow \sim(Q \rightarrow P)$

(100) $P \vee Q$; $P \rightarrow \sim Q$ / $(P \,\&\, \sim Q) \vee (Q \,\&\, \sim P)$

(101) $(P \vee Q) \rightarrow (P \,\&\, Q)$ / $(\sim P \vee \sim Q) \rightarrow (\sim P \,\&\, \sim Q)$

(102) $P \,\&\, (Q \vee R)$ / $(P \,\&\, Q) \vee (P \,\&\, R)$

(103) $(P \,\&\, Q) \vee (P \,\&\, R)$ / $P \,\&\, (Q \vee R)$

(104) $P \vee (Q \,\&\, R)$ / $(P \vee Q) \,\&\, (P \vee R)$

(105) $(P \,\&\, Q) \vee [(P \,\&\, R) \vee (Q \,\&\, R)]$ / $P \vee (Q \,\&\, R)$

(106) $P \vee Q$; $P \vee R$; $Q \vee R$ / $[P \,\&\, Q] \vee [(P \,\&\, R) \vee (Q \,\&\, R)]$

(107) $(P \rightarrow Q) \vee (P \rightarrow R)$ / $P \rightarrow (Q \vee R)$

(108) $(P \rightarrow R) \vee (Q \rightarrow R)$ / $(P \,\&\, Q) \rightarrow R$

(109) $P \leftrightarrow (Q \,\&\, \sim P)$ / $\sim(P \vee Q)$

(110) $(P \,\&\, Q) \vee (\sim P \,\&\, \sim Q)$ / $P \leftrightarrow Q$

EXERCISE SET H (Miscellaneous)

(111) $P \rightarrow (Q \vee R)$ / $(P \rightarrow Q) \vee (P \rightarrow R)$

(112) $(P \leftrightarrow Q) \rightarrow R$ / $P \rightarrow (Q \rightarrow R)$

(113) $P \rightarrow (\sim Q \rightarrow R)$ / $\sim(P \rightarrow R) \rightarrow Q$

(114) $(P \,\&\, Q) \rightarrow R$ / $(P \rightarrow R) \vee (Q \rightarrow R)$

(115) $P \leftrightarrow \sim Q$ / $(P \,\&\, \sim Q) \vee (Q \,\&\, \sim P)$

(116) $(P \rightarrow \sim Q) \rightarrow R$ / $\sim(P \,\&\, Q) \rightarrow R$

(117) $P \leftrightarrow (Q \,\&\, \sim P)$ / $\sim P \,\&\, \sim Q$

(118) P / $(P \,\&\, Q) \vee (P \,\&\, \sim Q)$

(119) $P \leftrightarrow \sim P$ / Q

(120) $(P \leftrightarrow Q) \leftrightarrow R$ / $P \leftrightarrow (Q \leftrightarrow R)$

23. ANSWERS TO EXERCISES FOR CHAPTER 5

EXERCISE SET A

#1:

(1)	P	Pr
(2)	P → Q	Pr
(3)	Q → R	Pr
(4)	R → S	Pr
(5)	Q	1,2,MP
(6)	R	3,5,MP
(7)	S	4,6,MP

#2:

(1)	P → Q	Pr
(2)	Q → R	Pr
(3)	R → S	Pr
(4)	~S	Pr
(5)	~R	3,4,MT
(6)	~Q	2,5,MT
(7)	~P	1,6,MT

#3:

(1)	~P ∨ Q	Pr
(2)	~Q	Pr
(3)	P ∨ R	Pr
(4)	~P	1,2,MTP2
(5)	R	3,4,MTP1

#4:

(1)	P ∨ Q	Pr
(2)	~P	Pr
(3)	Q → R	Pr
(4)	Q	1,2,MTP1
(5)	R	3,4,MP

#5:

(1)	P	Pr
(2)	P → ~Q	Pr
(3)	R → Q	Pr
(4)	~R → S	Pr
(5)	~Q	1,2,MP
(6)	~R	3,5,MT
(7)	S	4,6,MP

#6:

(1)	P ∨ ~Q	Pr
(2)	~P	Pr
(3)	R → Q	Pr
(4)	~R → S	Pr
(5)	~Q	1,2,MTP1
(6)	~R	3,5,MT
(7)	S	4,6,MP

#7:

(1)	(P → Q) → P	Pr
(2)	P → Q	Pr
(3)	P	1,2,MP
(4)	Q	2,3,MP

#8:

(1)	(P → Q) → R	Pr
(2)	R → P	Pr
(3)	P → Q	Pr
(4)	R	1,3,MP
(5)	P	2,4,MP
(6)	Q	3,5,MP

#9:

(1)	(P → Q) → (Q → R)	Pr
(2)	P → Q	Pr
(3)	P	Pr
(4)	Q → R	1,2,MP
(5)	Q	2,3,MP
(6)	R	4,5,MP

#10:

(1)	~P → Q	Pr
(2)	~Q	Pr
(3)	R ∨ ~P	Pr
(4)	~~P	1,2,MT
(5)	R	3,4,MTP2

#11:

(1)	~P → (~Q ∨ R)	Pr
(2)	P → R	Pr
(3)	~R	Pr
(4)	~P	2,3,MT
(5)	~Q ∨ R	1,4,MP
(6)	~Q	3,5,MTP2

#12:

(1)	P → ~Q	Pr
(2)	~S → P	Pr
(3)	~~Q	Pr
(4)	~P	1,3,MT
(5)	~~S	2,4,MT

#13:

(1)	P ∨ Q	Pr
(2)	Q → R	Pr
(3)	~R	Pr
(4)	~Q	2,3,MT
(5)	P	1,4,MTP2

#14:

(1)	~P → (Q ∨ R)	Pr
(2)	P → Q	Pr
(3)	~Q	Pr
(4)	~P	2,3,MT
(5)	Q ∨ R	1,4,MP
(6)	R	3,5,MTP1

#15:

(1)	P → R	Pr
(2)	~P → (S ∨ R)	Pr
(3)	~R	Pr
(4)	~P	1,3,MT
(5)	S ∨ R	2,4,MP
(6)	S	3,6,MTP2

#16:

(1)	P ∨ ~Q	Pr
(2)	~R → ~~Q	Pr
(3)	R → ~S	Pr
(4)	~~S	Pr
(5)	~R	3,4,MT
(6)	~~Q	2,5,MP
(7)	P	1,6,MTP2

#17:

(1)	(P → Q) ∨ (R → S)	Pr
(2)	(P → Q) → R	Pr
(3)	~R	Pr
(4)	~(P → Q)	2,3,MT
(5)	R → S	1,4,MTP1

#18:

(1)	(P → Q) → (R → S)	Pr
(2)	(R → T) ∨ (P → Q)	Pr
(3)	~(R → T)	Pr
(4)	P → Q	2,3,MTP1
(5)	R → S	1,4,MP

#19:

(1)	~R → (P ∨ Q)	Pr
(2)	R → P	Pr
(3)	(R → P) → ~P	Pr
(4)	~P	2,3,MP
(5)	~R	2,4,MT
(6)	P ∨ Q	1,5,MP
(7)	Q	4,6,MTP1

#20:

(1)	(P → Q) ∨ R	Pr
(2)	[(P → Q) ∨ R] → ~R	Pr
(3)	(P → Q) → (Q → R)	Pr
(4)	~R	1,2,MP
(5)	P → Q	1,4,MTP2
(6)	Q → R	3,5,MP
(7)	~Q	4,6,MT

EXERCISE SETS B-H

#1:

(1)	P	Pr
(2)	P → Q	Pr
(3)	Q → R	Pr
(4)	R → S	Pr
(5)	~~SHOW~~: S	DD
(6)	| Q	1,2,→O
(7)	| R	3,6,→O
(8)	| S	4,7,→O

#2:

(1)	P → Q	Pr
(2)	Q → R	Pr
(3)	R → S	Pr
(4)	~S	Pr
(5)	~~SHOW~~: ~P	DD
(6)	| ~R	3,4,→O
(7)	| ~Q	2,6,→O
(8)	| ~P	1,7,→O

#3:

(1)	~P ∨ Q	Pr
(2)	~Q	Pr
(3)	P ∨ R	Pr
(4)	~~SHOW~~: R	DD
(5)	| ~P	1,2,∨O
(6)	| R	3,5,∨O

#4:

(1)	P ∨ Q	Pr
(2)	~P	Pr
(3)	Q → R	Pr
(4)	~~SHOW~~: R	DD
(5)	| Q	1,2,∨O
(6)	| R	3,5,→O

#5:

(1)	P	Pr
(2)	P → ~Q	Pr
(3)	R → Q	Pr
(4)	~R → S	Pr
(5)	~~SHOW~~: S	DD
(6)	| ~Q	1,2,→O
(7)	| ~R	3,6,→O
(8)	| S	4,7,→O

#6:

(1)	P ∨ ~Q	Pr
(2)	~P	Pr
(3)	R → Q	Pr
(4)	~R → S	Pr
(5)	~~SHOW~~: S	DD
(6)	~Q	1,2,∨O
(7)	~R	3,6,→O
(8)	S	4,7,→O

#7:

(1)	(P → Q) → P	Pr
(2)	P → Q	Pr
(3)	~~SHOW~~: Q	DD
(4)	P	1,2,→O
(5)	Q	2,4,→O

#8:

(1)	(P → Q) → R	Pr
(2)	R → P	Pr
(3)	P → Q	Pr
(4)	~~SHOW~~: Q	DD
(5)	R	1,3,→O
(6)	P	2,5,→O
(7)	Q	3,6,→O

#9:

(1)	(P → Q) → (Q → R)	Pr
(2)	P → Q	Pr
(3)	P	Pr
(4)	~~SHOW~~: R	DD
(5)	Q → R	1,2,→O
(6)	Q	2,3,→O
(7)	R	5,6,→O

#10:

(1)	~P → Q	Pr
(2)	~Q	Pr
(3)	R ∨ ~P	Pr
(4)	~~SHOW~~: R	DD
(5)	~~P	1,2,→O
(6)	R	3,5,∨O

#11:

(1)	~P → (~Q ∨ R)	Pr
(2)	P → R	Pr
(3)	~R	Pr
(4)	~~SHOW~~: ~Q	DD
(5)	~P	2,3,→O
(6)	~Q ∨ R	1,5,→O
(7)	~Q	3,6,∨O

#12:

(1)	P → ~Q	Pr
(2)	~S → P	Pr
(3)	~~Q	Pr
(4)	~~SHOW~~: ~~S	DD
(5)	~P	1,3,→O
(6)	~~S	2,5,→O

#13:

(1)	P ∨ Q	Pr
(2)	Q → R	Pr
(3)	~R	Pr
(4)	~~SHOW~~: P	DD
(5)	~Q	2,3,→O
(6)	P	1,5,∨O

#14:

(1)	~P → (Q ∨ R)	Pr
(2)	P → Q	Pr
(3)	~Q	Pr
(4)	~~SHOW~~: R	DD
(5)	~P	2,3,→O
(6)	Q ∨ R	1,5,→O
(7)	R	3,6,∨O

#15:

(1)	P → R	Pr
(2)	~P → (S ∨ R)	Pr
(3)	~R	Pr
(4)	~~SHOW~~: S	DD
(5)	~P	1,3,→O
(6)	S ∨ R	2,5,→O
(7)	S	3,6,∨O

#16:

(1)	P ∨ ~Q	Pr
(2)	~R → ~~Q	Pr
(3)	R → ~S	Pr
(4)	~~S	Pr
(5)	~~SHOW~~: P	DD
(6)	~R	3,4,→O
(7)	~~Q	2,6,→O
(8)	P	1,7,∨O

#17:

(1)	(P → Q) ∨ (R → S)	Pr
(2)	(P → Q) → R	Pr
(3)	~R	Pr
(4)	~~SHOW~~: R → S	DD
(5)	~(P → Q)	2,3,→O
(6)	R → S	1,5,∨O

#18:

(1)	(P → Q) → (R → S)	Pr
(2)	(R → T) ∨ (P → Q)	Pr
(3)	~(R → T)	Pr
(4)	~~SHOW~~: R → S	DD
(5)	P → Q	2,3,∨O
(6)	R → S	1,5,→O

#19:

(1)	~R → (P ∨ Q)	Pr
(2)	R → P	Pr
(3)	(R → P) → ~P	Pr
(4)	~~SHOW~~: Q	DD
(5)	~P	2,3,→O
(6)	~R	2,5,→O
(7)	P ∨ Q	1,6,→O
(8)	Q	5,7,∨O

#20:

(1)	(P → Q) ∨ R	Pr
(2)	[(P → Q) ∨ R] → ~R	Pr
(3)	(P → Q) → (Q → R)	Pr
(4)	~~SHOW~~: ~Q	DD
(5)	~R	1,2,→O
(6)	P → Q	1,5,∨O
(7)	Q → R	3,6,→O
(8)	~Q	5,7,→O

#21:

(1)	P & Q	Pr
(2)	P → (R & S)	Pr
(3)	~~SHOW~~: Q & S	DD
(4)	P	1,&O
(5)	Q	1,&O
(6)	R & S	2,4,→O
(7)	S	6,&O
(8)	Q & S	5,7,&I

#22:

(1)	P & Q	Pr
(2)	(P ∨ R) → S	Pr
(3)	~~SHOW~~: P & S	DD
(4)	P	1,&O
(5)	P ∨ R	4,∨I
(6)	S	2,5,→O
(7)	P & S	4,6,&I

#23:

(1)	P	Pr
(2)	(P ∨ Q) → (R & S)	Pr
(3)	(R ∨ T) → U	Pr
(4)	~~SHOW~~: U	DD
(5)	P ∨ Q	1,∨I
(6)	R & S	2,5,→O
(7)	R	6,&O
(8)	R ∨ T	7,∨I
(9)	U	3,8,→O

#24:

(1)	P → Q	Pr
(2)	P ∨ R	Pr
(3)	~Q	Pr
(4)	~~SHOW~~: R & ~P	DD
(5)	~P	1,3,→O
(6)	R	2,5,∨O
(7)	R & ~P	5,6,&I

#25:

(1)	P → Q	Pr
(2)	~R → (Q → S)	Pr
(3)	R → T	Pr
(4)	~T & P	Pr
(5)	~~SHOW~~: Q & S	DD
(6)	~T	4,&O
(7)	~R	3,6,→O
(8)	Q → S	2,7,→O
(9)	P	4,&O
(10)	Q	1,9,→O
(11)	S	8,10:→O
(12)	Q & S	10,11,&I

#26:

(1)	P → Q	Pr
(2)	R ∨ ~Q	Pr
(3)	~R & S	Pr
(4)	(~P & S) → T	Pr
(5)	~~SHOW~~: T	DD
(6)	~R	3,&O
(7)	S	3,&O
(8)	~Q	2,6,∨O
(9)	~P	1,8,→O
(10)	~P & S	7,9,&I
(11)	T	4,10,→O

#27:

(1)	P ∨ ~Q	Pr
(2)	~R → Q	Pr
(3)	R → ~S	Pr
(4)	S	Pr
(5)	~~SHOW~~: P	DD
(6)	~~S	4,DN
(7)	~R	3,6,→O
(8)	Q	2,7,→O
(9)	~~Q	8,DN
(10)	P	1,9,∨O

#28:

(1)	P & Q	Pr
(2)	(P ∨ T) → R	Pr
(3)	S → ~R	Pr
(4)	~~SHOW~~: ~S	DD
(5)	P	1,&O
(6)	P ∨ T	5,∨I
(7)	R	2,6,→O
(8)	~~R	7,DN
(9)	~S	3,8,→O

#29:

(1)	P & Q	Pr
(2)	P → R	Pr
(3)	(P & R) → S	Pr
(4)	~~SHOW~~: Q & S	DD
(5)	P	1,&O
(6)	R	2,5,→O
(7)	P & R	5,6,&I
(8)	S	3,7,→O
(9)	Q	1,&O
(10)	Q & S	8,9,&I

#30:

(1)	P → Q	Pr
(2)	Q ∨ R	Pr
(3)	(R & ~P) → S	Pr
(4)	~Q	Pr
(5)	~~SHOW~~: S	DD
(6)	~P	1,4,→O
(7)	R	2,4,∨O
(8)	R & ~P	6,7,&I
(9)	S	3,8,→O

#31:

(1)	P & Q	Pr
(2)	~~SHOW~~: Q & P	DD
(3)	P	1,&O
(4)	Q	1,&O
(5)	Q & P	3,4,&I

#32:

(1)	P & (Q & R)	Pr
(2)	~~SHOW~~: (P & Q) & R	DD
(3)	P	1,&O
(4)	Q & R	1,&O
(5)	Q	4,&O
(6)	P & Q	3,5,&I
(7)	R	4,&O
(8)	(P & Q) & R	6,7,&I

#33:

(1)	P	Pr
(2)	~~SHOW~~: P & P	DD
(3)	P & P	1,1,&I

#34:

(1)	P	Pr
(2)	~~SHOW~~: P & (P ∨ Q)	DD
(3)	P ∨ Q	1,∨I
(4)	P & (P ∨ Q)	1,3,&I

#35:

(1)	P & ~P	Pr
(2)	~~SHOW~~: Q	DD
(3)	P	1,&O
(4)	~P	1,&O
(5)	P ∨ Q	3,∨I
(6)	Q	4,5,∨O

#36:

(1)	P ↔ ~Q	Pr
(2)	Q	Pr
(3)	P ↔ ~S	Pr
(4)	~~SHOW~~: S	DD
(5)	P → ~Q	1,↔O
(6)	~~Q	2,DN
(7)	~P	5,6,→O
(8)	~S → P	3,↔O
(9)	~~S	7,8,→O
(10)	S	9,DN

#37:

(1)	P & ~Q	Pr
(2)	Q ∨ (P → S)	Pr
(3)	(R & T) ↔ S	Pr
(4)	~~SHOW~~: P & R	DD
(5)	P	1,&O
(6)	~Q	1,&O
(7)	P → S	2,6,∨O
(8)	S	5,7,→O
(9)	S → (R & T)	3,↔O
(10)	R & T	8,9,→O
(11)	R	10:&O
(12)	P & R	5,11,&I

#38:

(1)	P → Q	Pr
(2)	(P → Q) → (Q → P)	Pr
(3)	(P ↔ Q) → P	Pr
(4)	~~SHOW~~: P & Q	DD
(5)	│ Q → P	1,2,→O
(6)	│ P ↔ Q	1,5,↔I
(7)	│ P	3,6,→O
(8)	│ Q	1,7,→O
(9)	│ P & Q	7,8,&I

#39:

(1)	~P & Q	Pr
(2)	(R ∨ Q) → (~S → P)	Pr
(3)	~S ↔ T	Pr
(4)	~~SHOW~~: ~T	DD
(5)	│ Q	1,&O
(6)	│ R ∨ Q	5,∨I
(7)	│ ~S → P	2,6,→O
(8)	│ ~P	1,&O
(9)	│ ~~S	7,8,→O
(10)	│ T → ~S	3,↔O
(11)	│ ~T	9,10,→O

#40:

(1)	P & ~Q	Pr
(2)	Q ∨ (R → S)	Pr
(3)	~V → ~P	Pr
(4)	V → (S → R)	Pr
(5)	(R ↔ S) → T	Pr
(6)	U ↔ (~Q & T)	Pr
(7)	~~SHOW~~: U	DD
(8)	│ P	1,&O
(9)	│ ~~P	8,DN
(10)	│ ~~V	3,9,→O
(11)	│ V	10,DN
(12)	│ S → R	4,11,→O
(13)	│ ~Q	1,&O
(14)	│ R → S	2,13,∨O
(15)	│ R ↔ S	12,14,↔I
(16)	│ T	5,15,→O
(17)	│ ~Q & T	13,16,&I
(18)	│ (~Q & T) → U	6,↔O
(19)	│ U	17,18,→O

#41:

(1)	(P ∨ Q) → R	Pr
(2)	~~SHOW~~: Q → R	CD
(3)	│ Q	As
(4)	│ ~~SHOW~~: R	DD
(5)	│ │ P ∨ Q	3,∨I
(6)	│ │ R	1,5,→O

#42:

(1)	Q → R	Pr
(2)	~~SHOW~~: (P & Q) → (P & R)	CD
(3)	│ P & Q	As
(4)	│ ~~SHOW~~: P & R	DD
(5)	│ │ P	3,&O
(6)	│ │ Q	3,&O
(7)	│ │ R	1,6,→O
(8)	│ │ P & R	5,7,&I

#43:

(1)	P → Q	Pr
(2)	~~SHOW~~: (Q → R) → (P → R)	CD
(3)	│ Q → R	As
(4)	│ ~~SHOW~~: P → R	CD
(5)	│ │ P	As
(6)	│ │ ~~SHOW~~: R	DD
(7)	│ │ │ Q	1,5,→O
(8)	│ │ │ R	3,7,→O

#44:

(1)	P → Q	Pr
(2)	~~SHOW~~: (R → P) → (R → Q)	CD
(3)	│ R → P	As
(4)	│ ~~SHOW~~: R → Q	CD
(5)	│ │ R	As
(6)	│ │ ~~SHOW~~: Q	DD
(7)	│ │ │ P	3,5,→O
(8)	│ │ │ Q	1,7,→O

#45:

(1)	(P & Q) → R	Pr
(2)	~~SHOW~~: P → (Q → R)	CD
(3)	│ P	As
(4)	│ ~~SHOW~~: Q → R	CD
(5)	│ │ Q	As
(6)	│ │ ~~SHOW~~: R	DD
(7)	│ │ │ P & Q	3,5,&I
(8)	│ │ │ R	1,7,→O

#46:

(1)	P → (Q → R)	Pr
(2)	~~SHOW~~: (P → Q) → (P → R)	CD
(3)	│ P → Q	As
(4)	│ ~~SHOW~~: P → R	CD
(5)	│ │ P	As
(6)	│ │ ~~SHOW~~: R	DD
(7)	│ │ │ Q	3,5,→O
(8)	│ │ │ Q → R	1,5,→O
(9)	│ │ │ R	7,8,→O

#47:

(1)	(P & Q) → R	Pr
(2)	~~SHOW~~: [(P→Q)→P]→[(P→Q)→R]	CD
(3)	│ (P → Q) → P	As
(4)	│ ~~SHOW~~: (P → Q) → R	CD
(5)	│ │ P → Q	As
(6)	│ │ ~~SHOW~~: R	DD
(7)	│ │ │ P	3,5,→O
(8)	│ │ │ Q	5,7,→O
(9)	│ │ │ P & Q	7,8,&I
(10)	│ │ │ R	1,9,→O

#48:

(1)	(P & Q) → (R → S)	Pr
(2)	~~SHOW~~: (P → Q) → [(P & R) → S]	CD
(3)	│ P → Q	As
(4)	│ ~~SHOW~~: (P & R) → S	CD
(5)	│ │ P & R	As
(6)	│ │ ~~SHOW~~: S	DD
(7)	│ │ │ P	5,&O
(8)	│ │ │ Q	3,7,→O
(9)	│ │ │ P & Q	7,8,&I
(10)	│ │ │ R → S	1,9,→O
(11)	│ │ │ R	5,&O
(12)	│ │ │ S	10:11→O

#49:

(1)	[(P & Q) & R] → S	Pr
(2)	~~SHOW~~: P → [Q → (R → S)]	CD
(3)	│ P	As
(4)	│ ~~SHOW~~: Q → (R → S)	CD
(5)	│ │ Q	As
(6)	│ │ ~~SHOW~~: R → S	CD
(7)	│ │ │ R	As
(8)	│ │ │ ~~SHOW~~: S	DD
(9)	│ │ │ │ P & Q	3,5,&I
(10)	│ │ │ │ (P & Q) & R	7,9,&I
(11)	│ │ │ │ S	1,10,→O

#50:

(1)	(~P & Q) → R	Pr
(2)	~~SHOW~~: (~Q → P) → (~P → R)	CD
(3)	│ ~Q → P	As
(4)	│ ~~SHOW~~: ~P → R	CD
(5)	│ │ ~P	As
(6)	│ │ ~~SHOW~~: R	DD
(7)	│ │ │ ~~Q	3,5,→O
(8)	│ │ │ Q	7,DN
(9)	│ │ │ ~P & Q	5,8,&I
(10)	│ │ │ R	1,9,→O

#51:

(1)	P → Q	Pr
(2)	P → ~Q	Pr
(3)	~~SHOW~~: ~P	ID
(4)	│ P	As
(5)	│ ~~SHOW~~: ※	DD
(6)	│ │ Q	1,4,→O
(7)	│ │ ~Q	2,4,→O
(8)	│ │ ※	6,7,※I

#52:

(1)	P → Q	Pr
(2)	Q → ~P	Pr
(3)	~~SHOW~~: ~P	ID
(4)	│ P	As
(5)	│ ~~SHOW~~: ※	DD
(6)	│ │ Q	1,4,→O
(7)	│ │ ~~P	4,DN
(8)	│ │ ~Q	2,7,→O
(9)	│ │ ※	6,8,※I

#53:

(1)	P → Q	Pr
(2)	~Q ∨ ~R	Pr
(3)	P → R	Pr
(4)	~~SHOW~~: ~P	ID
(5)	│ P	As
(6)	│ ~~SHOW~~: ※	DD
(7)	│ │ Q	1,5,→O
(8)	│ │ ~~Q	7,DN
(9)	│ │ ~R	2,8,∨O
(10)	│ │ ~P	3,9:→O
(11)	│ │ ※	5,10,※I

#54:

(1)	P → R	Pr
(2)	Q → ~R	Pr
(3)	~~SHOW~~: ~(P & Q)	ID
(4)	│ P & Q	As
(5)	│ ~~SHOW~~: ※	DD
(6)	│ │ P	4,&O
(7)	│ │ Q	4,&O
(8)	│ │ R	1,6,→O
(9)	│ │ ~R	2,7,→O
(10)	│ │ ※	8,9,※I

#55:

(1)	P & Q	Pr
(2)	~~SHOW~~: ~(P → ~Q)	ID
(3)	│ P → ~Q	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ P	1,&O
(6)	│ │ Q	1,&O
(7)	│ │ ~Q	3,5,→O
(8)	│ │ ※	6,7,※I

#56:

(1)	P & ~Q	Pr
(2)	~~SHOW~~: ~(P → Q)	ID
(3)	│ P → Q	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ P	1,&O
(6)	│ │ ~Q	1,&O
(7)	│ │ Q	3,5,→O
(8)	│ │ ※	6,7,※I

#57:

(1)	~P	Pr
(2)	~~SHOW~~: ~(P & Q)	ID
(3)	│ P & Q	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ P	3,&O
(6)	│ │ ※	1,5,※I

#58:

(1)	~P & ~Q	Pr
(2)	~~SHOW~~: ~(P ∨ Q)	ID
(3)	│ P ∨ Q	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ ~P	1,&O
(6)	│ │ ~Q	1,&O
(7)	│ │ Q	3,5,∨O
(8)	│ │ ※	6,7,※I

#59:

(1)	P ↔ Q	Pr
(2)	~Q	Pr
(3)	~~SHOW~~: ~(P ∨ Q)	ID
(4)	│ P ∨ Q	As
(5)	│ ~~SHOW~~: ※	DD
(6)	│ │ P	2,4,∨O
(7)	│ │ P → Q	1,↔O
(8)	│ │ Q	6,7,→O
(9)	│ │ ※	2,8,※I

#60:

(1)	P & Q	Pr
(2)	~~SHOW~~: ~(~P ∨ ~Q)	ID
(3)	│ ~P ∨ ~Q	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ P	1,&O
(6)	│ │ Q	1,&O
(7)	│ │ ~~P	5,DN
(8)	│ │ ~Q	3,7,∨O
(9)	│ │ ※	6,8,※I

#61:

(1)	~P ∨ ~Q	Pr
(2)	~~SHOW~~: ~(P & Q)	ID
(3)	│ P & Q	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ P	3,&O
(6)	│ │ Q	3:&O
(7)	│ │ ~~P	5,DN
(8)	│ │ ~Q	1,7,∨O
(9)	│ │ ※	6,8,※I

#62:

(1)	P ∨ Q	Pr
(2)	~~SHOW~~: ~(~P & ~Q)	ID
(3)	│ ~P & ~Q	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ ~P	3,&O
(6)	│ │ ~Q	3,&O
(7)	│ │ Q	1,5,∨O
(8)	│ │ ※	6,7,※I

#63:

(1)	P → Q	Pr
(2)	~~SHOW~~: ~(P & ~Q)	ID
(3)	│ P & ~Q	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ P	3,&O
(6)	│ │ ~Q	3,&O
(7)	│ │ Q	1,5,→O
(8)	│ │ ※	6,7,※I

#64:

(1)	P → (Q → ~P)	Pr
(2)	~~SHOW~~: P → ~Q	CD
(3)	│ P	As
(4)	│ ~~SHOW~~: ~Q	ID
(5)	│ │ Q	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ Q → ~P	1,3,→O
(8)	│ │ ~P	5,7,→O
(9)	│ │ ※	3,8,※I

#65:

(1)	(P & Q) → R	Pr
(2)	~~SHOW~~: (P & ~R) → ~Q	CD
(3)	│ P & ~R	As
(4)	│ ~~SHOW~~: ~Q	ID
(5)	│ │ Q	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ │ P	3,&O
(8)	│ │ │ P & Q	5,7,&I
(9)	│ │ │ R	1,8,→O
(10)	│ │ │ ~R	3,&O
(11)	│ │ │ ※	9,10,※I

#66:

(1)	(P & Q) → ~R	Pr
(2)	~~SHOW~~: P → ~(Q & R)	CD
(3)	P	As
(4)	~~SHOW~~: ~(Q & R)	ID
(5)	Q & R	As
(6)	~~SHOW~~: ※	DD
(7)	Q	5,&O
(8)	P & Q	3,7,&I
(9)	~R	1,8,→O
(10)	R	5,&O
(11)	※	9,10,※I

#67:

(1)	P → (Q → R)	Pr
(2)	~~SHOW~~: (Q & ~R) → ~P	CD
(3)	Q & ~R	As
(4)	~~SHOW~~: ~P	ID
(5)	P	As
(6)	~~SHOW~~: ※	DD
(7)	Q → R	1,5,→O
(8)	Q	3,&O
(9)	R	7,8,→O
(10)	~R	3,&O
(11)	※	9,10,※I

#68:

(1)	P → ~(Q & R)	Pr
(2)	~~SHOW~~: (P & Q) → ~R	CD
(3)	P & Q	As
(4)	~~SHOW~~: ~R	ID
(5)	R	As
(6)	~~SHOW~~: ※	DD
(7)	P	3,&O
(8)	Q	3,&O
(9)	Q & R	5,8,&I
(10)	~(Q & R)	1,7,→O
(11)	※	9:10,※I

#69:

(1)	P → ~(Q & R)	Pr
(2)	~~SHOW~~: (P → Q) → (P → ~R)	CD
(3)	P → Q	As
(4)	~~SHOW~~: P → ~R	CD
(5)	P	As
(6)	~~SHOW~~: ~R	ID
(7)	R	As
(8)	~~SHOW~~: ※	DD
(9)	Q	3,5,→O
(10)	Q & R	7,9,&I
(11)	~(Q & R)	1,5,→O
(12)	※	10,11,※I

#70:

(1)	P → (Q → R)	Pr
(2)	~~SHOW~~: (P → ~R) → (P → ~Q)	CD
(3)	P → ~R	As
(4)	~~SHOW~~: P → ~Q	CD
(5)	P	As
(6)	~~SHOW~~: ~Q	ID
(7)	Q	As
(8)	~~SHOW~~: ※	DD
(9)	Q → R	1,5,→O
(10)	~R	3,5,→O
(11)	~Q	9,10,→O
(12)	※	7,11,※I

#71:

(1)	P → Q	Pr
(2)	~P → Q	Pr
(3)	~~SHOW~~: Q	ID
(4)	~Q	As
(5)	~~SHOW~~: ※	DD
(6)	~P	1,4,→O
(7)	~~P	2,4,→O
(8)	※	6,7,※I

#72:

(1)	P ∨ Q	Pr
(2)	P → R	Pr
(3)	Q ∨ ~R	Pr
(4)	~~SHOW~~: Q	ID
(5)	~Q	As
(6)	~~SHOW~~: ※	DD
(7)	P	1,5,∨O
(8)	R	2,7,→O
(9)	~R	3,5,∨O
(10)	※	8,9,※I

#73:

(1)	~P → R	Pr
(2)	Q → R	Pr
(3)	P → Q	Pr
(4)	~~SHOW~~: R	ID
(5)	~R	As
(6)	~~SHOW~~: ※	DD
(7)	~Q	2,5,→O
(8)	~~P	1,5,→O
(9)	P	8,DN
(10)	Q	3,9,→O
(11)	※	7,10,※I

#74:

(1)	(P ∨ ~Q) → (R & ~S)	Pr
(2)	Q ∨ S	Pr
(3)	~~SHOW~~: Q	ID
(4)	│ ~Q	As
(5)	│ ~~SHOW~~: ※	DD
(6)	│ │ P ∨ ~Q	4,∨I
(7)	│ │ R & ~S	1,6,→O
(8)	│ │ ~S	7,&O
(9)	│ │ S	2,4,∨O
(10)	│ │ ※	8,9,※I

#75:

(1)	(P ∨ Q) → (R → S)	Pr
(2)	(~S ∨ T) → (P & R)	Pr
(3)	~~SHOW~~: S	ID
(4)	│ ~S	As
(5)	│ ~~SHOW~~: ※	DD
(6)	│ │ ~S ∨ T	4,∨I
(7)	│ │ P & R	2,6,→O
(8)	│ │ P	7,&O
(9)	│ │ P ∨ Q	8,∨I
(10)	│ │ R → S	1,9,→O
(11)	│ │ R	7,&O
(12)	│ │ S	10,11,→O
(13)	│ │ ※	4,12,※I

#76:

(1)	~(P & ~Q)	Pr
(2)	~~SHOW~~: P → Q	CD
(3)	│ P	As
(4)	│ ~~SHOW~~: Q	ID
(5)	│ │ ~Q	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ │ P & ~Q	3,5,&I
(8)	│ │ │ ※	1,7,※I

#77:

(1)	P → (~Q → R)	Pr
(2)	~~SHOW~~: (P & ~R) → Q	CD
(3)	│ P & ~R	As
(4)	│ ~~SHOW~~: Q	ID
(5)	│ │ ~Q	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ P	3,&O
(8)	│ │ ~R	3,&O
(9)	│ │ ~Q → R	1,7,→O
(10)	│ │ ~~Q	8,9,→O
(11)	│ │ ※	5,10,※I

#78:

(1)	P & (Q ∨ R)	Pr
(2)	~~SHOW~~: ~(P & Q) → R	CD
(3)	│ ~(P & Q)	As
(4)	│ ~~SHOW~~: R	ID
(5)	│ │ ~R	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ │ Q ∨ R	1,&O
(8)	│ │ │ Q	5,7,∨O
(9)	│ │ │ P	1,&O
(10)	│ │ │ P & Q	8,9,&I
(11)	│ │ │ ※	3,10,※I

#79:

(1)	P ∨ Q	Pr
(2)	~~SHOW~~: Q ∨ P	ID
(3)	│ ~(Q ∨ P)	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ ~Q	3,~∨O
(6)	│ │ ~P	3,~∨O
(7)	│ │ Q	1,6,∨O
(8)	│ │ ※	5,7,※I

#80:

(1)	~P → Q	Pr
(2)	~~SHOW~~: P ∨ Q	ID
(3)	│ ~(P ∨ Q)	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ │ ~P	3,~∨O
(6)	│ │ │ ~Q	3,~∨O
(7)	│ │ │ Q	1,5,→O
(8)	│ │ │ ※	6,7,※I

#81:

(1)	~(P & Q)	Pr
(2)	~~SHOW~~: ~P ∨ ~Q	ID
(3)	│ ~(~P ∨ ~Q)	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ ~~P	3,~∨O
(6)	│ │ ~~Q	3,~∨O
(7)	│ │ P	5,DN
(8)	│ │ Q	6,DN
(9)	│ │ P & Q	7,8,&I
(10)	│ │ ※	1,9,※I

#82:

(1)	P → Q	Pr
(2)	~~SHOW~~: ~P ∨ Q	ID
(3)	│ ~(~P ∨ Q)	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ ~~P	3,~∨O
(6)	│ │ ~Q	3,~∨O
(7)	│ │ P	5,DN
(8)	│ │ Q	1,7,→O
(9)	│ │ ※	6,8,※I

#83:

(1)	P ∨ Q	Pr
(2)	P → R	Pr
(3)	Q → S	Pr
(4)	SHOW: R ∨ S	ID
(5)	~(R ∨ S)	As
(6)	SHOW: ※	DD
(7)	~R	5,~∨O
(8)	~S	5,~∨O
(9)	~P	2,7,→O
(10)	~Q	3,8,→O
(11)	Q	1,9,∨O
(12)	※	10,11,※I

#84:

(1)	~P → Q	Pr
(2)	P → R	Pr
(3)	SHOW: Q ∨ R	ID
(4)	~(Q ∨ R)	As
(5)	SHOW: ※	DD
(6)	~Q	4,~∨O
(7)	~R	4,~∨O
(8)	~~P	1,6,→O
(9)	P	8,DN
(10)	R	2,9,→O
(11)	※	7,10,※I

#85:

(1)	~P → Q	Pr
(2)	~R → S	Pr
(3)	~Q ∨ ~S	Pr
(4)	SHOW: P ∨ R	ID
(5)	~(P ∨ R)	As
(6)	SHOW: ※	DD
(7)	~P	5,~∨O
(8)	~R	5,~∨O
(9)	Q	1,7,→O
(10)	S	2,8,→O
(11)	~~Q	9,DN
(12)	~S	3,11,∨O
(13)	※	10,12,※I

#86:

(1)	(P & ~Q) → R	Pr
(2)	SHOW: P → (Q ∨ R)	CD
(3)	P	As
(4)	SHOW: Q ∨ R	ID
(5)	~(Q ∨ R)	As
(6)	SHOW: ※	DD
(7)	~Q	5,~∨O
(8)	P & ~Q	3,7,&I
(9)	R	1,8,→O
(10)	~R	5,~∨O
(11)	※	9,10,※I

#87

(1)	~P → (~Q ∨ R)	Pr
(2)	SHOW: Q → (P ∨ R)	CD
(3)	Q	As
(4)	SHOW: P ∨ R	ID
(5)	~(P ∨ R)	As
(6)	SHOW: ※	DD
(7)	~P	5,~∨O
(8)	~R	5,~∨O
(9)	~Q ∨ R	1,7,→O
(10)	~Q	8,9,∨O
(11)	※	3,10,※I

#88:

(1)	P & (Q ∨ R)	Pr
(2)	SHOW: (P & Q) ∨ R	ID
(3)	~[(P & Q) ∨ R]	As
(4)	SHOW: ※	DD
(5)	~(P & Q)	3,~∨O
(6)	~R	3,~∨O
(7)	P	1,&O
(8)	Q ∨ R	1,&O
(9)	Q	6,8,∨O
(10)	P & Q	7,9,&I
(11)	※	5,10,※I

#89:

(1)	(P ∨ Q) & (P ∨ R)	Pr
(2)	SHOW: P ∨ (Q & R)	ID
(3)	~[P ∨ (Q & R)]	As
(4)	SHOW: ※	DD
(5)	~P	3,~∨O
(6)	~(Q & R)	3,~∨O
(7)	P ∨ Q	1,&O
(8)	Q	5,7,∨O
(9)	P ∨ R	1,&O
(10)	R	5,9,∨O
(11)	Q & R	8,10,&I
(12)	※	6,11※I

#90:

(1)	(P ∨ Q) → (P & Q)	Pr
(2)	SHOW: (P&Q) ∨ (~P & ~ Q)	ID
(3)	~[(P & Q) ∨ (~P & ~Q)]	As
(4)	SHOW: ※	DD
(5)	~(P & Q)	3,~∨O
(6)	~(~P & ~Q)	3,~∨O
(7)	~(P ∨ Q)	1,5,→O
(8)	~P	7,~∨O
(9)	~Q	7,~∨O
(10)	~P & ~Q	8,9,&I
(11)	※	6,10,※I

#91:

(1)	P → (Q & R)	Pr
(2)	~~SHOW~~: (P → Q) & (P → R)	DD
(3)	│ ~~SHOW~~: P → Q	CD
(4)	│ │ P	As
(5)	│ │ ~~SHOW~~: Q	DD
(6)	│ │ │ Q & R	1,4,→O
(7)	│ │ │ Q	6,&O
(8)	│ ~~SHOW~~: P → R	CD
(9)	│ │ P	As
(10)	│ │ ~~SHOW~~: R	DD
(11)	│ │ │ Q & R	1,9,→O
(12)	│ │ │ R	11&O
(13)	│ (P → Q) & (P → R)	3,8,&I

#92:

(1)	(P ∨ Q) → R	Pr
(2)	~~SHOW~~: (P → R) & (Q → R)	DD
(3)	│ ~~SHOW~~: P → R	CD
(4)	│ │ P	As
(5)	│ │ ~~SHOW~~: R	DD
(6)	│ │ │ P ∨ Q	4,∨I
(7)	│ │ │ R	1,6,→O
(8)	│ ~~SHOW~~: Q → R	CD
(9)	│ │ Q	As
(10)	│ │ ~~SHOW~~: R	DD
(11)	│ │ │ P ∨ Q	9,∨I
(12)	│ │ │ R	1,11,→O
(13)	│ (P → R) & (Q → R)	3,8,&I

#93:

(1)	(P ∨ Q) → (P & Q)	Pr
(2)	~~SHOW~~: P ↔ Q	DD
(3)	│ ~~SHOW~~: P → Q	CD
(4)	│ │ P	As
(5)	│ │ ~~SHOW~~: Q	DD
(6)	│ │ │ P ∨ Q	4,∨I
(7)	│ │ │ P & Q	1,6,→O
(8)	│ │ │ Q	7,&O
(9)	│ ~~SHOW~~: Q → P	CD
(10)	│ │ Q	As
(11)	│ │ ~~SHOW~~: P	DD
(12)	│ │ │ P ∨ Q	10,∨I
(13)	│ │ │ P & Q	1,12,→O
(14)	│ │ │ P	13,&O
(15)	│ P ↔ Q	3,9,↔I

#94:

(1)	P ↔ Q	Pr
(2)	~~SHOW~~: Q ↔ P	DD
(3)	│ P → Q	1,↔O
(4)	│ Q → P	1,↔O
(5)	│ Q ↔ P	3,4,↔I

#95:

(1)	P ↔ Q	Pr
(2)	~~SHOW~~: ~P ↔ ~Q	DD
(3)	│ ~~SHOW~~: ~P → ~Q	CD
(4)	│ │ ~P	As
(5)	│ │ ~~SHOW~~: ~Q	DD
(6)	│ │ │ Q → P	1,↔O
(7)	│ │ │ ~Q	4,6,→O
(8)	│ ~~SHOW~~: ~Q → ~P	CD
(9)	│ │ ~Q	As
(10)	│ │ ~~SHOW~~: ~P	DD
(11)	│ │ │ P → Q	1,↔O
(12)	│ │ │ ~P	9,11,→O
(13)	│ ~P ↔ ~Q	3,8,↔I

#96:

(1)	P ↔ Q	Pr
(2)	Q → ~P	Pr
(3)	~~SHOW~~: ~P & ~Q	DD
(4)	│ ~~SHOW~~: ~P	ID
(5)	│ │ P	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ │ P → Q	1,↔O
(8)	│ │ │ Q	5,7,→O
(9)	│ │ │ ~P	2,8,→O
(10)	│ │ │ ※	5,9,※I
(11)	│ ~~SHOW~~: ~Q	ID
(12)	│ │ Q	As
(13)	│ │ ~~SHOW~~: ※	DD
(14)	│ │ │ Q → P	1,↔O
(15)	│ │ │ P	12,14,→O
(16)	│ │ │ ~P	2,12,→O
(17)	│ │ │ ※	15,16,※I
(18)	│ ~P & ~Q	4,11,&I

#97:

(1)	(P → Q) ∨ (~Q → R)	Pr
(2)	~~SHOW~~: P → (Q ∨ R)	CD
(3)	│ P	As
(4)	│ ~~SHOW~~: Q ∨ R	ID
(5)	│ │ ~(Q ∨ R)	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ │ ~Q	5,~∨O
(8)	│ │ │ ~R	5,~∨O
(9)	│ │ │ ~~SHOW~~: ~(P → Q)	ID
(10)	│ │ │ │ P → Q	As
(11)	│ │ │ │ ~~SHOW~~: ※	DD
(12)	│ │ │ │ │ Q	3,10,→O
(13)	│ │ │ │ │ ※	7,12,※I
(14)	│ │ │ ~Q → R	1,9,∨O
(15)	│ │ │ R	7,14,→O
(16)	│ │ │ ※	8,15,※I

#98:

(1)	P ∨ Q	Pr
(2)	P → ~Q	Pr
(3)	~~SHOW~~: (P → Q) → (Q & ~P)	CD
(4)	P → Q	As
(5)	~~SHOW~~: Q & ~P	DD
(6)	~~SHOW~~: Q	ID
(7)	~Q	As
(8)	~~SHOW~~: ※	DD
(9)	~P	4,7,→O
(10)	P	1,7,∨O
(11)	※	9,10,※I
(12)	~~SHOW~~: ~P	ID
(13)	P	As
(14)	~~SHOW~~: ※	DD
(15)	Q	4,13,→O
(16)	~Q	2,13,→O
(17)	※	15,16,※I
(18)	Q & ~P	6,12,&I

#99:

(1)	P ∨ Q	Pr
(2)	~(P & Q)	Pr
(3)	~~SHOW~~: (P → Q) → ~(Q → P)	CD
(4)	P → Q	As
(5)	~~SHOW~~: ~(Q → P)	ID
(6)	Q → P	As
(7)	~~SHOW~~: ※	DD
(8)	P → ~Q	2,~&O
(9)	~~SHOW~~: ~P	ID
(10)	P	As
(11)	~~SHOW~~: ※	DD
(12)	Q	4,10,→O
(13)	~Q	8,10,→O
(14)	※	12,13,※I
(15)	Q	1,9,∨O
(16)	P	6,15,→O
(17)	※	9,16,※I

#100:

(1)	P ∨ Q	Pr
(2)	P → ~Q	Pr
(3)	~~SHOW~~: (P & ~Q) ∨ (Q & ~P)	ID
(4)	~[(P & ~Q) ∨ (Q & ~P)]	As
(5)	~~SHOW~~: ※	DD
(6)	~(P & ~Q)	4,~∨O
(7)	~(Q & ~P)	4,~∨O
(8)	~~SHOW~~: ~P	ID
(9)	P	As
(10)	~~SHOW~~: ※	DD
(11)	~Q	2,9,→O
(12)	P & ~Q	9,11,&I
(13)	※	6,12,※I
(14)	Q	1,8,∨O
(15)	Q & ~P	8,14,&I
(16)	※	7,15,※I

#101:

(1)	(P ∨ Q) → (P & Q)	Pr
(2)	~~SHOW~~: (~P ∨ ~Q) → (~P & ~Q)	CD
(3)	~P ∨ ~Q	As
(4)	~~SHOW~~: ~P & ~Q	DD
(5)	~~SHOW~~: ~P	ID
(6)	P	As
(7)	~~SHOW~~: ※	DD
(8)	P ∨ Q	6,∨I
(9)	P & Q	1,8,→O
(10)	~~P	6,DN
(11)	~Q	3,10,∨O
(12)	Q	9,&O
(13)	※	11,12,※I
(14)	~~SHOW~~: ~Q	ID
(15)	Q	As
(16)	~~SHOW~~: ※	DD
(17)	P ∨ Q	15,∨I
(18)	P & Q	1,17,→O
(19)	~~Q	15,DN
(20)	~P	3,19,∨O
(21)	P	18,&O
(22)	※	20,21,※I
(23)	~P & ~Q	5,14,&I

#102:

(1)	P & (Q ∨ R)	Pr
(2)	~~SHOW~~: (P & Q) ∨ (P & R)	ID
(3)	~[(P & Q) ∨ (P & R)]	As
(4)	~~SHOW~~: ※	DD
(5)	~(P & Q)	3,~∨O
(6)	~(P & R)	3,~∨O
(7)	P	1,&O
(8)	Q ∨ R	1,&O
(9)	P → ~Q	5,~&O
(10)	P → ~R	6,~&O
(11)	~Q	7,9,→O
(12)	R	8,11,∨O
(13)	~R	7,10,→O
(14)	※	12,13,※I

#103:

(1)	(P & Q) ∨ (P & R)	Pr
(2)	~~SHOW~~: P & (Q ∨ R)	DD
(3)	~~SHOW~~: P	ID
(4)	~P	As
(5)	~~SHOW~~: ※	DD
(6)	~~SHOW~~: ~(P & Q)	ID
(7)	P & Q	As
(8)	~~SHOW~~: ※	DD
(9)	P	7,&O
(10)	※	4,9,※I
(13)	P & R	1,6,∨O
(14)	P	13,&O
(15)	※	4,14,※I
(16)	~~SHOW~~: Q ∨ R	ID
(17)	~(Q ∨ R)	As
(18)	~~SHOW~~: ※	DD
(19)	~Q	17,~∨O
(20)	~R	17,~∨O
(21)	~~SHOW~~: ~(P & Q)	ID
(22)	P & Q	As
(23)	~~SHOW~~: ※	DD
(24)	Q	22,&O
(25)	※	19.24,※I

#104:

(1)	P ∨ (Q & R)	Pr
(2)	~~SHOW~~: (P ∨ Q) & (P ∨ R)	DD
(3)	~~SHOW~~: P ∨ Q	ID
(4)	~(P ∨ Q)	As
(5)	~~SHOW~~: ※	DD
(6)	~P	4,~∨O
(7)	~Q	4,~∨O
(8)	Q & R	1,6,∨O
(9)	Q	8,&O
(10)	※	7,9,※I
(11)	~~SHOW~~: P ∨ R	ID
(12)	~(P ∨ R)	As
(13)	~~SHOW~~: ※	DD
(14)	~P	12,~∨O
(15)	~R	12,~∨O
(16)	Q & R	1,14,∨O
(17)	Q	16,&O
(18)	※	15,17,※I
(19)	(P ∨ Q) & (P ∨ R)	3,11,&I

#105:

(1)	(P&Q) ∨ [(P&R) ∨ (Q&R)]	Pr
(2)	~~SHOW~~: P ∨ (Q & R)	ID
(3)	~[P ∨ (Q & R)]	As
(4)	~~SHOW~~: ※	DD
(5)	~P	3,~∨O
(6)	~(Q & R)	3,~∨O
(7)	~~SHOW~~: ~(P & Q)	ID
(8)	P & Q	As
(9)	~~SHOW~~: ※	DD
(10)	P	8,&O
(11)	※	5,10,※I
(12)	(P & R) ∨ (Q & R)	1,7,∨O
(13)	P & R	6,12,∨O
(14)	P	13,&O
(15)	※	5,14,※I

#106:

(1)	P ∨ Q	Pr
(2)	P ∨ R	Pr
(3)	Q ∨ R	Pr
(4)	~~SHOW~~: (P&Q)∨[(P&R)∨(Q&R)]	ID
(5)	~{(P&Q)∨[(P&R)∨(Q&R)]}	As
(6)	~~SHOW~~: ※	DD
(7)	~(P & Q)	5,~∨O
(8)	~[(P & R) ∨ (Q & R)]	5,~∨O
(8)	~(P & R)	8,~∨O
(9)	~(Q & R)	8,~∨O
(10)	P → ~Q	7,~&O
(11)	P → ~R	8,~&O
(12)	Q → ~R	9,~&O
(13)	~~SHOW~~: ~P	ID
(14)	P	As
(15)	~~SHOW~~: ※	DD
(16)	~Q	10,14,→O
(17)	~R	11,14,→O
(18)	R	3,16,∨O
(19)	※	17,18,※I
(20)	Q	1,13,∨O
(21)	R	2,13,∨O
(22)	~R	12,20,→O
(23)	※	21,22,※I

#107:

(1)	(P → Q) ∨ (P → R)	Pr
(2)	~~SHOW~~: P → (Q ∨ R)	CD
(3)	P	As
(4)	~~SHOW~~: Q ∨ R	ID
(5)	~(Q ∨ R)	As
(6)	~~SHOW~~: ※	DD
(7)	~Q	5,~∨O
(8)	~R	5,~∨O
(9)	~~SHOW~~: ~(P → Q)	ID
(10)	P → Q	As
(11)	~~SHOW~~: ※	DD
(12)	Q	3,10,→O
(13)	※	7,12,※I
(14)	P → R	1,9 ∨O
(15)	R	3,14,→O
(16)	※	8,15,※I

#108:

(1)	(P → R) ∨ (Q → R)	Pr
(2)	~~SHOW~~: (P & Q) → R	CD
(3)	P & Q	As
(4)	~~SHOW~~: R	ID
(5)	~R	As
(6)	~~SHOW~~: ※	DD
(7)	~~SHOW~~: ~(P → R)	ID
(8)	P → R	As
(9)	~~SHOW~~: ※	DD
(10)	P	3,&O
(11)	R	8,10,→O
(12)	※	5,11,※I
(13)	Q → R	1,7,∨O
(14)	Q	3,&O
(15)	R	13,14,→O
(16)	※	5,15,※I

#109:

(1)	P ↔ (Q & ~P)	Pr
(2)	~~SHOW~~: ~(P ∨ Q)	ID
(3)	P ∨ Q	As
(4)	~~SHOW~~: ※	DD
(5)	P → (Q & ~P)	1,↔O
(6)	~~SHOW~~: P	ID
(7)	~P	As
(8)	~~SHOW~~: ※	DD
(9)	Q	3,7,∨O
(10)	Q & ~P	7,9,&I
(11)	(Q & ~P) → P	1,→O
(12)	P	10,12,→O
(13)	※	7,12,※I
(14)	Q & ~P	5,6,→O
(15)	~P	14,&O
(16)	※	6,15,※I

#110:

(1)	(P & Q) ∨ (~P & ~Q)	Pr
(2)	~~SHOW~~: P ↔ Q	DD
(3)	~~SHOW~~: P → Q	CD
(4)	P	As
(5)	~~SHOW~~: Q	ID
(6)	~Q	As
(7)	~~SHOW~~: ※	DD
(8)	~~SHOW~~: ~(P & Q)	ID
(9)	P & Q	As
(10)	~~SHOW~~: ※	DD
(11)	Q	9,&O
(12)	※	6,11,※I
(13)	~P & ~Q	1,8,∨O
(14)	~P	13,&O
(15)	※	4,14,※I
(16)	~~SHOW~~: Q → P	CD
(17)	Q	As
(18)	~~SHOW~~: P	ID
(19)	~P	As
(20)	~~SHOW~~: ※	DD
(21)	~~SHOW~~: ~(P & Q)	ID
(22)	P & Q	As
(23)	~~SHOW~~: ※	DD
(24)	P	22,&O
(25)	※	19,24,※I
(26)	~P & ~Q	1,21,&O
(27)	~Q	26,&O
(28)	※	17,27,※I
(29)	P ↔ Q	3,16,↔I

#111:

(1)	P → (Q ∨ R)	Pr
(2)	~~SHOW~~: (P → Q) ∨ (P → R)	ID
(3)	~[(P → Q) ∨ (P → R)]	As
(4)	~~SHOW~~: ※	DD
(5)	~(P → Q)	3,~∨O
(6)	~(P → R)	3,~∨O
(7)	P & ~Q	5,~→O
(8)	P & ~R	6,~→O
(9)	P	7,&O
(10)	~Q	7,&O
(11)	~R	8,&O
(12)	Q ∨ R	1,9,→O
(13)	R	10,12,∨O
(14)	※	11,13,※I

#112:

(1)	(P ↔ Q) → R	Pr
(2)	~~SHOW~~: P → (Q → R)	CD
(3)	│ P	As
(4)	│ ~~SHOW~~: Q → R	CD
(5)	│ │ Q	As
(6)	│ │ ~~SHOW~~: R	DD
(7)	│ │ │ ~R	As
(8)	│ │ │ ~~SHOW~~: ※	DD
(9)	│ │ │ │ ~(P ↔ Q)	1,7,→O
(10)	│ │ │ │ ~P ↔ Q	9,~↔O
(11)	│ │ │ │ Q → ~P	9,↔O
(12)	│ │ │ │ ~P	5,11,→O
(13)	│ │ │ │ ※	3,12,※I

#113:

(1)	P → (~Q → R)	Pr
(2)	~~SHOW~~: ~(P → R) → Q	CD
(3)	│ ~(P → R)	As
(4)	│ ~~SHOW~~: Q	ID
(5)	│ │ ~Q	As
(6)	│ │ ~~SHOW~~: ※	DD
(7)	│ │ │ P & ~R	3,~→O
(8)	│ │ │ P	7,&O
(9)	│ │ │ ~R	7,&O
(10)	│ │ │ ~Q → R	1,8,→O
(11)	│ │ │ R	5,10,→O
(12)	│ │ │ ※	9,11,※I

#114:

(1)	(P & Q) → R	Pr
(2)	~~SHOW~~: (P → R) ∨ (Q → R)	ID
(3)	│ ~[(P → R) ∨ (Q → R)]	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ ~(P → R)	3,~∨O
(6)	│ │ ~(Q → R)	3,~∨O
(7)	│ │ P & ~R	5,~→O
(8)	│ │ Q & ~R	6,~→O
(9)	│ │ P	7,&O
(10)	│ │ ~R	7,&O
(11)	│ │ Q	7,&O
(12)	│ │ P & Q	9,11,&I
(13)	│ │ R	1,12,→O
(14)	│ │ ※	10,13,※I

#115:

(1)	P ↔ ~Q	Pr
(2)	~~SHOW~~: (P & ~Q) ∨ (Q & ~P)	ID
(3)	│ ~[(P & ~Q) ∨ (Q & ~P)]	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ ~(P & ~Q)	3,~∨O
(6)	│ │ ~(Q & ~P)	3,~∨O
(7)	│ │ P → ~~Q	5,~&O
(8)	│ │ Q → ~~P	6,~&O
(9)	│ │ P → ~Q	1,↔O
(10)	│ │ ~Q → P	1,↔O
(11)	│ │ ~~SHOW~~: P	ID
(12)	│ │ │ ~P	As
(13)	│ │ │ ~~SHOW~~: ※	DD
(14)	│ │ │ │ ~~Q	7,10,→O
(15)	│ │ │ │ ~~~P	12,DN
(16)	│ │ │ │ ~Q	7,15,→O
(17)	│ │ │ │ ※	14,16,※I
(18)	│ │ ~~Q	7,11,→O
(19)	│ │ ~Q	9,11,→O
(20)	│ │ ※	18,19,※I

#116:

(1)	(P → ~Q) → R	Pr
(2)	~~SHOW~~: ~(P & Q) → R	CD
(3)	│ ~(P & Q)	As
(4)	│ ~~SHOW~~: R	DD
(5)	│ │ P → ~Q	3,~&O
(6)	│ │ R	1,5,→O

#117:

(1)	P ↔ (Q & ~P)	Pr
(2)	~~SHOW~~: ~P & ~Q	DD
(3)	│ ~~SHOW~~: ~P	ID
(4)	│ │ P	As
(5)	│ │ ~~SHOW~~: ※	DD
(6)	│ │ │ P → (Q & ~P)	1,↔O
(7)	│ │ │ Q & ~P	4,6,→O
(8)	│ │ │ ~P	7,&O
(9)	│ │ │ ※	4,8,※I
(10)	│ ~~SHOW~~: ~Q	ID
(11)	│ │ Q	As
(12)	│ │ ~~SHOW~~: ※	DD
(13)	│ │ │ Q & ~P	3,11,&I
(14)	│ │ │ (Q & ~P) → P	1,↔O
(15)	│ │ │ P	13,14,→O
(16)	│ │ │ ※	3,15,※I
(17)	│ ~P & ~Q	3,10,&I

#118:

(1)	P	Pr
(2)	~~SHOW~~: (P & Q) ∨ (P & ~Q)	ID
(3)	│ ~[(P & Q) ∨ (P & ~Q)]	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ ~(P & Q)	3,~∨O
(6)	│ │ ~(P & ~Q)	3,~∨O
(7)	│ │ P → ~Q	5,~&O
(8)	│ │ P → ~~Q	6,~&O
(9)	│ │ ~Q	1,7,→O
(10)	│ │ ~~Q	1,8,→O
(11)	│ │ ※	9,10,※I

#119:

(1)	P ↔ ~P	Pr
(2)	~~SHOW~~: Q	ID
(3)	│ ~Q	As
(4)	│ ~~SHOW~~: ※	DD
(5)	│ │ P → ~P	1,↔O
(6)	│ │ ~P → P	1,↔O
(7)	│ │ ~~SHOW~~: P	ID
(8)	│ │ │ ~P	As
(9)	│ │ │ ~~SHOW~~: ※	DD
(10)	│ │ │ │ P	6,8,→O
(11)	│ │ │ │ ※	8,10,※I
(12)	│ │ ~P	5,7,→O
(13)	│ │ ※	7,12,※I

#120:

(1)	(P ↔ Q) ↔ R	Pr
(2)	~~SHOW~~: P ↔ (Q ↔ R)	DD
(3)	│ ~~SHOW~~: P → (Q ↔ R)	CD
(4)	│ │ P	As
(5)	│ │ ~~SHOW~~: Q ↔ R	DD
(6)	│ │ │ ~~SHOW~~: Q → R	CD
(7)	│ │ │ │ Q	As
(8)	│ │ │ │ ~~SHOW~~: R	DD
(9)	│ │ │ │ │ ~~SHOW~~: P → Q	CD
(10)	│ │ │ │ │ │ P	As
(11)	│ │ │ │ │ │ ~~SHOW~~: Q	DD
(12)	│ │ │ │ │ │ │ Q	7,R
(13)	│ │ │ │ │ ~~SHOW~~: Q → P	CD
(14)	│ │ │ │ │ │ Q	As
(15)	│ │ │ │ │ │ ~~SHOW~~: P	DD
(16)	│ │ │ │ │ │ │ P	4,R
(17)	│ │ │ │ │ P ↔ Q	9,13,↔I
(18)	│ │ │ │ │ (P ↔ Q) → R	1,↔O
(19)	│ │ │ │ │ R	17,18,→O
(20)	│ │ │ ~~SHOW~~: R → Q	CD
(21)	│ │ │ │ R	As
(22)	│ │ │ │ ~~SHOW~~: Q	DD
(23)	│ │ │ │ │ R → (P ↔ Q)	1,↔O
(24)	│ │ │ │ │ P ↔ Q	21,23→O
(25)	│ │ │ │ │ P → Q	24,↔O
(26)	│ │ │ │ │ Q	4,25,→O
(27)	│ │ │ Q ↔ R	6,20,↔I
(28)	│ ~~SHOW~~: (Q ↔ R) → P	CD
(29)	│ │ Q ↔ R	As
(30)	│ │ ~~SHOW~~: P	ID
(31)	│ │ │ ~P	As
(32)	│ │ │ ~~SHOW~~: ※	DD
(33)	│ │ │ │ ~~SHOW~~: P → Q	CD
(34)	│ │ │ │ │ P	As
(35)	│ │ │ │ │ ~~SHOW~~: Q	ID
(36)	│ │ │ │ │ │ ~Q	As
(37)	│ │ │ │ │ │ ~~SHOW~~: ※	DD
(38)	│ │ │ │ │ │ │ ※	31,34,※I
(39)	│ │ │ │ ~~SHOW~~: Q → P	CD
(40)	│ │ │ │ │ Q	As
(41)	│ │ │ │ │ ~~SHOW~~: P	DD
(42)	│ │ │ │ │ │ Q → R	29,↔O
(43)	│ │ │ │ │ │ R	40,42,→O
(44)	│ │ │ │ │ │ R → (P ↔ Q)	1,↔O
(45)	│ │ │ │ │ │ P ↔ Q	43,44,→O
(46)	│ │ │ │ │ │ Q → P	45,↔O
(47)	│ │ │ │ │ │ P	40,46,→O
(48)	│ │ │ │ P ↔ Q	33,39,↔I
(49)	│ │ │ │ (P ↔ Q) → R	1,↔O
(50)	│ │ │ │ R	48,49,→O
(51)	│ │ │ │ R → Q	29,↔O
(52)	│ │ │ │ Q	50,51,→O
(53)	│ │ │ │ P	39,52,→O
(54)	│ │ │ │ ※	31,53,※I
(55)	│ P ↔ (Q ↔ R)	3,28,↔I