6 TRANSLATIONS IN MONADIC PREDICATE LOGIC

1. INTRODUCTION

As we have noted in earlier chapters, the validity of an argument is a function of its form, as opposed to its specific content. On the other hand, as we have also noted, the form of a statement or an argument is not absolute, but rather depends upon the level of logical analysis we are pursuing.

We have already considered two levels of logical analysis – syllogistic logic, and sentential logic. Whereas syllogistic logic considers quantifier expressions (e.g., 'all', 'some') as the sole logical terms, sentential logic considers statement connectives (e.g., 'and', 'or') as the sole logical connectives. Thus, these branches of logic analyze logical form quite differently from one another.

Predicate logic subsumes both syllogistic logic and sentential logic; in particular, it considers both quantifier expressions and statement connectives as logical terms. It accordingly represents a deeper level of logical analysis. As a consequence of the deeper logical analysis, numerous arguments that are not valid, either relative to syllogistic logic, or relative to sentential logic, turn out to be valid relative to predicate logic. Consider the following argument.

(A) if at least one person will show up, then we will meet
Adams will show up
/ we will meet

First of all, argument (A) is not a syllogism, so it is not a valid syllogism. Next, if we symbolize (A) in sentential logic, we obtain something like the following.

(F) $P \rightarrow M$
A
$/ M$

Here 'P' stands for 'at least one person will show up', 'A' stands for 'Adams will show up', and 'M' stands for 'we will meet'. It is easy to show (using truth tables) that (F) is not a valid sentential logic form.

Nevertheless, argument (A) is valid (intuitively, at least). What this means is that the formal techniques of sentential logic are not fully adequate to characterize the validity of arguments. In particular, (A) has further logical structure that is not captured by sentential logic. So, what we need is a further technique for uncovering the additional structure of (A) that reveals that it is indeed valid. This technique is provided by predicate logic.

2. THE SUBJECT-PREDICATE FORM OF ATOMIC STATEMENTS

Recall the distinction in sentential logic between the following sentences.

(1) Jay and Kay are Sophomores
(2) Jay and Kay are roommates

Whereas the former is equivalent to a conjunction, namely,

(1*) Jay is a Sophomore and Kay is a Sophomore,

the latter is an atomic statement, having no structure from the viewpoint of sentential logic. In particular, whereas in (1) 'and' is used *conjunctively* to assert something about Jay and Kay individually, in (2) 'and' is used *relationally* – to assert that a certain relation holds between Jay and Kay.

In predicate logic, we are able to uncover the additional logical structure of (2); indeed, we are able to uncover the additional logical structure of (1) as well. In particular, we are able to display atomic formulas as consisting of a predicate and one or more subjects.

Consider the atomic statements that compose (1).

(3) Jay is a Sophomore
(4) Kay is a Sophomore

Each of these consists of two grammatical components: a *subject* and a *predicate*. In (3), the subject is 'Jay', and the predicate is '...is a sophomore'; in (4), the subject is 'Kay', and the predicate is the same, '...is a sophomore'.

Next, consider the sole atomic statement in (2), which is (2) itself.

(5) Jay and Kay are roommates

This may be paraphrased as follows.

(5*) Jay is a roommate of Kay

Unlike (3) and (4), this sentence has two grammatical subjects – 'Jay' and 'Kay'. In addition to the subjects, there is also a predicate – '...is a roommate of...'

The basic idea in the three examples so far is that an atomic sentence can be grammatically analyzed into a predicate and one or more subjects. In order to further emphasize this point, let us consider a slightly more complicated example, involving several subjects in addition to a single predicate.

(6) Chris is sitting between Jay and Kay

Once more 'and' is used relationally rather than conjunctively; in particular (6) is not a conjunction, but is rather atomic. In this case, the predicate is fairly complex:

...is sitting between...and...,

and there are three grammatical subjects:

Chris, Jay, Kay

We now state the first principle of predicate logic.

> In predicate logic, every atomic sentence consists of one *predicate* and one or more *subjects*.

3. PREDICATES

Every predicate has a degree, which is a number. If a predicate has degree one, we call it a *one-place predicate*; if it has degree two, we call it a *two-place predicate*; and so forth.

In principle, for every number n, there are predicates of degree n, (i.e., n-place predicates). However, we are going to concentrate primarily on l-place, 2-place, and 3-place predicates, in that order of emphasis.

To say that a predicate is a one-place predicate is to say that it takes a single grammatical subject. In other words, a one-place predicate forms a statement when combined with a *single* subject. The following are examples.

___ is clever
___ is a Sophomore
___ sleeps soundly
___ is very unhappy

Each of these is a l-place predicate, because it takes a single term to form a statement; thus, for example, we obtain the following statements.

<u>Jay</u> is clever
<u>Kay</u> is a Sophomore
<u>Chris</u> sleeps soundly
<u>Max</u> is very unhappy

On the other hand, a *two-place predicate* takes two grammatical subjects, which is to say that it forms a statement when combined with two names. The following are examples.

___ is taller than ___
___ is south of ___
___ admires ___
___ respects ___
___ is a cousin of ___

Thus, for example, using various pairs of individual names, we obtain the following statements.

Jones is taller than Smith
New York is south of Boston
Jay admires Kay
Kay respects Jay
Jay is a cousin of Kay

Finally, a *three-place predicate* takes three grammatical subjects, which is to say that it forms a statement when combined with three names. The following are examples.

___ is between ___ and ___
___ is a child of ___ and ___
___ is the sum of ___ and ___
___ borrowed ___ from ___
___ loaned ___ to ___
___ recommended ___ to ___

Thus, for example, we may obtain the following statements from these predicates.

New York is between Boston and Philadelphia
Chris is a child of Jay and Kay
11 is the sum of 4 and 7
Jay borrowed this pen from Kay
Kay loaned this pen to Jay
Kay recommended the movie "Casablanca" to Jay

One-place predicates (also called monadic predicates) may be thought of as denoting *properties* (e.g., the property of being tall), whereas multi-place (1,2,3-place) predicates (also called polyadic predicates) may be thought of as denoting *relations* (e.g., the relation between two things when one is taller than the other).

Sometimes, the study of predicate logic is formally divided into *monadic predicate logic* (also called property logic) and *polyadic predicate logic* (also called relational logic). In this text, we do not *formally* divide the subject in this way. On the other hand, we deal primarily with monadic predicate logic in the present chapter, leaving polyadic predicate logic for the next chapter.

4. SINGULAR TERMS

Predicate logic analyzes every atomic sentence into a predicate and one or more subjects. In the present section, we examine the latter in a little more detail. In the previous section, the alert reader probably noticed that diverse sorts of expressions were substituted into the blanks of the predicates. Not only did we use names of people, but we also used numerals (which are names of numbers), the name of a movie, and even a demonstrative noun phrase '*this* pen'.

These are all examples of *singular terms* (also called individual terms), which include four sorts of expressions, among others.

(1) proper nouns
(2) definite descriptions
(3) demonstrative noun phrases
(4) pronouns

Examples of proper nouns include the following.

Jay, Kay, Chris, etc.
George Washington, John F. Kennedy, etc.
Paris, London, New York, etc.
Jupiter, Mars, Venus, etc.
1, 2, 3, 23, 45, etc.

Examples of definite descriptions that are singular terms include the following.

the largest river in the world
James Joyce's last book
the president of the U.S.
the square root of 2
the first person to finish the exam

Examples of demonstrative noun phrases that are singular terms include the following.

the person over there
this person, this pen, etc.
that person, that pen, etc.

The use of demonstrative noun phrases generally involves *pointing*, either explicitly or implicitly.

Examples of pronouns that are singular terms are basically all the third person singular pronouns,

he, she, it, him, her,

as well as "wh" expressions such as

who, whom, which (that), what, when, where, why.

Having seen various examples of singular terms, it is equally important to see examples of noun-like expressions that do not qualify as singular terms. These might be called, by analogy, *plural terms*.

Examples of Plural Terms

the people who play for the New York Yankees
the five smartest persons in the class
James Joyce's books
the European cities
the natural numbers
the people standing over there
they, them, their, these, those

Note carefully, however, that ‘they’ and ‘them’, and ‘their’ are frequently used as singular pronouns, as in

(1) Every student likes *their* roommate.

In times long past, many writers thought that ‘he’, ‘him’, and ‘his’ could be used as singular third-person *neutral* pronouns, so that the following was the grammatically-proper formulation of (1).

(1') Every student likes *his* roommate.

However, in the latter third of the 20th Century, the neutrality of ‘he’/‘him’/‘his’ was generally rejected, and the following was substituted.

(1") Every student likes *his or her* roommate.

More recently still, the latter usage has been reconsidered in light of the growing number of individuals who do not identify with the traditional male/female binary classification, and choose the pronouns ‘they’/‘them’/‘their’. In that case, the more inclusive form would be:

(1"') Every student likes *his or her or their* roommate.

Although this is more inclusive, acknowledging non-binary students, it seems awkward, so most people opt for:

(1*) Every student likes *their* roommate,

which treats ‘their’ as a *neutral* third-person singular pronoun.

Note, however, that a small grammatical awkwardness attends this choice, since we are strongly inclined to use plural-marked verbs with ‘they’, as in:

(2) Every person **is** happy if they **are** virtuous.

as opposed to the following odd-sounding,

(2?) Every person **is** happy if they **is** virtuous.

But we have the very same problem with plural ‘you’ versus singular ‘you’. We do not say ‘you is’ even when ‘you’ refers to a single individual.

Now, back to singular terms. A singular term refers to a single individual – a person, place, thing, event, etc., although perhaps a complex one, like IBM, or a very complex one, like the Renaissance. In order to decide whether a noun phrase qualifies as a singular term, the simplest thing to do is to check whether the noun phrase can be used properly with the singular verb form ‘is’. Except for singular-‘you’ and singular-‘they’, if the noun phrase requires the plural form ‘are’, then it is not a singular term, but is rather a plural term.

We conclude by stating a further, very important, principle of the grammar of predicate logic.

> In predicate logic,
> every subject is a *singular term*.

5. ATOMIC FORMULAS

Having discussed the manner in which every atomic sentence of predicate logic is decomposed into a predicate and (singular) subject(s), we now introduce the symbolic apparatus by which the form of such a sentence is formally displayed.

In sentential logic, you will recall, atomic sentences are abbreviated by upper case letters of the Roman alphabet. The fact that they are symbolized by letters reflects the fact that they are regarded as having no further logical structure. By contrast, in predicate logic, every atomic sentence is analyzed into its constituents, being its predicate and its subject or subjects.

In order to distinguish these constituents, we adopt a particular notational convention, which is simple if not entirely intuitive. This convention is presented as follows.

> (1) Predicates are symbolized by upper case letters.
> (2) Singular terms are symbolized by lower case letters.
> (3) Every atomic sentence is symbolized by juxtaposing the associated subject and predicate letters.
> (4) In particular, in each atomic sentence, the predicate letter goes first and is followed by the subject letter(s).

The following are examples.

Expression	Abbreviation
Predicates:	
___ is tall	T
___ is a Freshman	F
___ respects____	R
___ is a cousin of___	C
___ is between ___ and ___	B
Singular Terms:	
Jay	j
Kay	k
New York City	n
Jupiter	j
the tallest person in class	t
the movie "Casablanca"	c
Sentences:	
Jay is tall	Tj
Kay is a Freshman	Fk
Jay respects Kay	Rjk
Kay is a cousin of Jay	Ckj
Chris is between Jay and Kay	Bcjk

From occasion to occasion, different predicates can be abbreviated by the same letter; likewise for singular terms. However, in any given context (a statement or argument), one must be careful to use different letters to abbreviate different names. The letter 'j' can stand for 'Jay' or for 'Jupiter', but if 'Jay' and 'Jupiter' appear in the same statement or argument, then we cannot use 'j' to abbreviate both of them; for example, we might use 'j' for 'Jay' and 'u' for 'Jupiter'.

Notice that we use lower case letters to abbreviate all singular terms, including definite descriptions. Unlike proper nouns, definite descriptions have further logical structure, and this further structure is revealed and examined in more advanced branches of logic. However, for the purposes of intro logic, definite descriptions have no further logical structure; they are simply singular terms, and are accordingly abbreviated simply by lower case letters.

6. VARIABLES AND PRONOUNS

So far we have concentrated on singular terms that might be called *constants*. In addition to constants there are also *variables*. Variables play the same role in predicate logic that (singular third-person) pronouns play in ordinary language; specifically, they are used for cross-referencing inside a sentence or larger linguistic unit. Furthermore, variables play the same role in predicate logic that variables play in symbolic arithmetic (called algebra in high school); specifically, they enable us to refer to individuals (e.g., individual numbers), without referring to any particular individual (number). This is very useful, as we shall see shortly, in making general claims.

Concerning symbolization, whereas we use the lower case letters ‘a’, ‘b’, ‘c’, ..., ‘w’ as constants, we use the remaining lower case letters ‘x, `y’, ‘z’ as variables. If it turns out that we need more than 26 constants or variables, then we will subscript these with numerals to obtain, for example, ‘a_2’, ‘y_3’, ‘z_{50}’, etc. Thus, in principle, there are infinitely many constants and variables.

In order to avoid using subscripted variables, we also reserve the right to "requisition" constants to use as variables, if the need should arise. So, for example, if we need six variables, but only a few constants, then we will "draft" ‘u’, ‘v’, and ‘w’ into service as variables. If this should happen, it will be explicitly announced. For the most part, however, in intro logic we need only three variables, and there is no need to recruit constants.

When we combine a predicate with one or more singular terms, we obtain a *formula* of predicate logic. When one or more of these singular terms is a variable, we obtain an *open formula*. Open formulas of predicate logic correspond to open sentences of natural language.

Consider the following sentences of arithmetic.

(1)	2 is even	Et
(2)	3 is larger than 4	Ltf
(3)	it is even	Ex
(4)	this is larger than that	Lxy

Whereas (1) and (2) are closed sentences, and their symbolizations, to the right, are closed formulas, (3) and (4) are open sentences, and their symbolizations are open formulas.

So, what is the difference between open and closed sentences, anyway? The difference can be described by saying that, whereas (1) and (2) express propositions and are accordingly true or false, (3) and (4) do not (by themselves) express propositions and are accordingly neither true nor false.

On the other hand (this is the tricky part!), even though it does not autonomously express a proposition, an open sentence can be *used* to assert a proposition – specifically, by uttering it while "pointing" at a particular object or objects. If we "point" at the number two (insofar as that is possible), and say “it/this/that is even”, then we have asserted the proposition that the number two is even; indeed, we have asserted a true proposition. Similarly, when we successively point at the number two and the number five, and say "this is larger than that", then we have asserted the proposition that two is larger than five; we have asserted a proposition, but a false proposition.

A closed sentence, by contrast, can be used to assert a proposition, even without having to point. If I say "two is even", I need not point at the number two in order to assert a proposition; the sentence does it for me.

One way to describe the difference between open and closed sentences is to say that, unlike closed sentences, open sentences are essentially *indexical* in character, which is to say that their use essentially involves *pointing*. (Here, think of the *index* finger, as used for pointing.) This pointing can be fairly straightforward, but it can also be oblique and subtle. This pointing can also be either *external* or *internal* to the sentence in which the indexical (i.e., pointing) expression occurs.

For example, in the sentence about the date with the music major, the pronoun refers to (points at) something external; the 'he or she' refers to the particular person about whom the speaker is talking. By contrast, in the sentence about roommates, the 'his or her' refers, not externally to a particular person, but rather internally to the expression 'everyone'.

Another use of internal pointing involves the following indexical expressions.

(1) the former
(2) the latter
(3) the party of the first part
(4) the party of the second part

The latter two expressions (an example of pointing!) are used almost exclusively in legal documents, and we will not examine them any further. The former two expressions, on the other hand, are important expressions in logic. If I refer to a music major and a business major, in that order, then if I say "the *former* respects the *latter*", I am saying that the music major respects the business major. If I say instead "he respects her", then it is not clear who respects whom. Thus, the words 'former' and 'latter' are useful substitutes for ordinary pronouns.

We conclude this section by announcing yet another principle of the grammar of predicate logic.

In an atomic formula,
every subject is either
a *constant* or a *variable*.

7. COMPOUND FORMULAS

We have now described the atomic formulas of predicate logic; every such formula consists of an n-place predicate letter followed by n singular terms, each one being either a constant or a variable. The atomic formulas of predicate logic play exactly the same role that atomic formulas play in sentential logic; in particular, they can be combined with connectives to form molecular formulas.

We already know how to construct molecular formulas from atomic formulas in sentential logic. This skill carries over directly to predicate logic, the rules being precisely the same. If we have a formula, we can form its negation; if we have two formulas, we can form their conjunction, disjunction, conditional, and biconditional. The only difference is that the simple statements we begin with are not simply letters, as in sentential logic, but are rather combinations of predicate letters and singular terms.

The following are examples of compound statements in predicate logic, followed by their symbolizations.

(1)	if Jay is a Freshman, then Kay is a Freshman	Fj → Fk
(2)	Kay is not a Freshman	~Fk
(3)	neither Jay nor Kay is a Freshman	~Fj & ~Fk
(4)	Jay respects Kay, but Kay does not respect Jay	Rjk & ~Rkj

Next, we note that either (or both) of the proper nouns 'Jay' and 'Kay' can be replaced by pronouns. Correspondingly, either (or both) of the constants 'j' and 'k' can be replaced by variables (for example, 'x' and 'y'). We accordingly obtain various open sentences (formulas). For example, taking (1), we can construct the following open statements and associated open formulas.

(1)	if Jay is a Freshman, then Kay is a Freshman	Fj → Fk
(1a)	if Jay is a Freshman, then she is a Freshman	Fj → Fy
(1b)	if he is a Freshman, then Kay is a Freshman	Fx → Fk
(1c)	if he is a Freshman, then she is a Freshman	Fx → Fy

8. QUANTIFIERS

We have already seen that compound formulas can be constructed using the connectives of sentential logic. In addition to these truth-functional connectives, predicate logic has additional compound forming expressions – namely, the quantifiers.

Quantifiers are linguistic expressions denoting *quantity* in some form. Examples of quantifiers in English include the following.

every, all, each, both, any, either
some, most, many, several, a few
none, neither
at least one, at least two, etc.
at most one, at most two, etc.
exactly one, exactly two, etc.

These expressions are typically combined with noun phrases to produce sentences, such as the following.

every Freshman is clever
at least one Sophomore is clever
no Senior is clever
many Sophomores are clever
several Juniors are clever

In addition to these quantifier expressions, there are also derivative expressions, contractions, involving ‘thing’ and ‘one’.

everyone, everything, someone, something, no one, nothing

These yield sentences such as the following

everyone is clever
everything is clever
someone is clever
something is clever
no one is clever
nothing is clever

Recall that there are numerous statement connectives in English, but in sentential logic we concentrate on just a few, logically fruitful, ones. Similarly, even though there are numerous quantifier expressions in English, in predicate logic we concentrate only on a couple of them, given as follows.

every
at least one

Not only do we concentrate on these two quantifier concepts, we render them very general, as follows.

everything is such that...
there is at least one thing such that...
at least one thing is such that...

Although these expressions are somewhat stilted (much like the official expression for negation ‘it is not true that...’), they are sufficiently general to be used in a much wider variety of contexts than more colloquial quantifier expressions.

If this is not stilted enough, we must add one further feature to the above quantifiers, in order to obtain the official quantifiers of predicate logic. Recall that a pronoun can point internally, and in particular, it can point at a quantifier expression in the sentence. In the sentence

everyone likes his/her roommate

the pronoun ‘his/her’ points at the quantifier ‘everyone’. But what if the sentence in question has more than one quantifier? Consider the following.

everyone knows someone who respects his/her mother

This sentence is ambiguous, because it isn't clear what the pronoun ‘his/her’ points at. This sentence might be paraphrased in either of the following ways.

everyone knows someone who respects *the former*'s mother
everyone knows someone who respects *the latter*'s mother

The additional feature needed by the quantifiers above is an *index*, in order to allow clear and consistent cross-referencing inside of sentences in which they appear. Since we are using variables as pronouns, it is convenient to use the very same symbolic devices as quantifier indices as well.

Thus, every quantifier comes with an index (a variable) attached to it. We thus obtain the following quantifier expressions.

everything x is such that...
everything y is such that...
everything z is such that...

there is at least one thing x such that...
there is at least one thing y such that...
there is at least one thing z such that...

These are symbolized respectively as follows.

$\forall x \quad \forall y \quad \forall z$
$\exists x \quad \exists y \quad \exists z$

Historically, the upside-down ‘A’ derives from the word ‘all’, and the backwards ‘E’ derives from the word ‘exist’. Whereas the expressions ‘$\forall x$’, ‘$\forall y$’, ‘$\forall z$’ are called *universal quantifiers*, the expressions ‘$\exists x$’, ‘$\exists y$’, ‘$\exists z$’ are called *existential quantifiers*.

For every variable, there are two quantifiers, a universal quantifier, and an existential quantifier. Grammatically, a quantifier is a one-place connective, just like negation $\sim$. In other words, we have the following grammatical principle.

If **F** is a formula, then so are all the following.

$\forall x\mathbf{F}$, $\forall y\mathbf{F}$, $\forall z\mathbf{F}$
$\exists x\mathbf{F}$, $\exists y\mathbf{F}$, $\exists z\mathbf{F}$

Of course, in forming the compound formula, the outer parentheses (if any) of the formula **F** must be restored before prefixing the quantifier. This is just like negation. We will see examples of this later.

We now have the official quantifier expressions of predicate logic. How do they combine with other formulas to make quantified formulas? The basic idea (but not the whole story) is that one begins with an open formula involving (say) the variable 'x', and one prefixes '∀x' to obtain a universally quantified formula, or one prefixes '∃x' to obtain an existentially quantified formula.

For example, we can begin with the following open formula,

Fx: x is fascinating (it is fascinating),

and prefix either '∀x' or '∃x' to obtain the following formulas.

∀xFx: everything [x] is such that
it [x] is fascinating

∃xFx: there is at least one thing [x] such that
it [x] is fascinating

In each case, I have divided the sentence into a quantifier and an open formula. The variables are placed in parentheses, since they are not really part of the English sentence; rather, they are used to cross-reference the pronoun 'it'. In particular, the fact that 'x' is used for both the quantifier and the pronoun indicates that 'it' points back at (cross-references) the quantifier expression.

This is the simplest case, one in which the open formula $\mathcal{A}$ is atomic. It can also be molecular; it can even be a quantified formula (a great deal more about this in the next chapter). The following are all examples of open formulas involving 'x' together with the resulting quantified formulas. Notice the appearance of the parentheses in (2) and (3).

	Open Formula:	**Universal Formula:**	**Existential Formula:**
(1)	~Fx	∀x~Fx	∃x~Fx
(2)	Fx & Gx	∀x(Fx & Gx)	∃x(Fx & Gx)
(3)	Fx → Gx	∀x(Fx → Gx)	∃x(Fx → Gx)
(4)	Rxj	∀xRxj	∃xRxj
(5)	∃yRxy	∀x∃yRxy	∃x∃yRxy

The pairs to the right are all examples of quantified formulas, universal formulas and existential formulas respectively. These can in turn be combined using any of the sentential logic connectives, to obtain (e.g.) the following compound formulas.

(6)	$\forall x{\sim}Fx \vee \forall x(Fx \,\&\, Gx)$	disjunction
(7)	${\sim}\forall xRxj$; ${\sim}\exists xRxj$; ${\sim}\forall x\exists yRxy$; ${\sim}\exists x\exists yRxy$	negations
(8)	$\forall xRxj \rightarrow \forall x\exists yRxy$; $\exists xRxj \rightarrow \exists x\exists yRxy$	conditionals

At this stage, the important thing is not necessarily to be able to read the above formulas, but to be able to recognize them as formulas. Toward this end, keep in clear sight the rules of formula formation in predicate logic, which are sketched as follows.

Definition of Formula in Predicate Logic:

Atomic Formulas:

(1) If P is a predicate letter of degree n, then P followed by n singular terms is an atomic formula.

(2) Nothing else is an atomic formula.

Formulas:

(1) Every atomic formula is a formula.

(2) If $\mathcal{A}$ is a formula, then so is ${\sim}\mathcal{A}$.

(3) If $\mathcal{A}$ and $\mathcal{B}$ are formulas then so are the following.

$(\mathcal{A} \,\&\, \mathcal{B})$
$(\mathcal{A} \vee \mathcal{B})$
$(\mathcal{A} \rightarrow \mathcal{B})$
$(\mathcal{A} \leftrightarrow \mathcal{B})$

(4) If $\mathcal{A}$ is a formula, then so are the following.

$\forall x\mathcal{A}$, $\forall y\mathcal{A}$, $\forall z\mathcal{A}$, etc.
$\exists x\mathcal{A}$, $\exists y\mathcal{A}$, $\exists z\mathcal{A}$, etc.

(5) Nothing else is a formula.

9. COMBINING QUANTIFIERS WITH NEGATION

As noted at the end of the previous section, any formula can be prefixed by either a universal quantifier or an existential quantifier, just as any formula can be prefixed by negation, and the result is another formula.

In the present section, we concentrate on the way in which negation interacts with quantifiers.

Let us start with the following open formula.

(1) Px it is perfect

Then let us quantify it both universally and existentially, as follows.

(2) $\forall x Px$ everything is such that
it is perfect

(3) $\exists x Px$ at least one thing is such that
it is perfect

These can in turn be negated, yielding the following formulas.

(4) $\sim\forall x Px$ it is not true that
everything is such that
it is perfect

(5) $\sim\exists x Px$ it is not true that
at least one thing is such that
it is perfect

Before considering more colloquial paraphrases of the above sentences, let us consider an alternative tack. Let us first negate ‘Px’ to obtain the following.

(6) $\sim Px$ it is not true that
it is perfect

The latter sentence may be paraphrased as either of the following.

it is not perfect
It is imperfect

Many adjectives have ready-made negations (happy/unhappy, friendly/unfriendly, possible/impossible); most adjectives, however, do not have natural negations. On the other hand, we can always produce the negation of any adjective simply by prefixing ‘non-’ in front of the adjective.

Now, let us take the negated formula ‘~Px’ and quantify in the two ways, which yields the following.

(7) $\forall x\sim Px$ everything is such that
it is not true that
it is perfect

everything is such that
it is not perfect

everything is such that
it is imperfect

(8) $\exists x\sim Px$ at least one thing is such that
it is not true that
it is perfect

at least one thing is such that
it is not perfect

at least one thing is such that
it is imperfect

Having written down all the simple formulas involving negation and quantifiers, let us now consider the idiomatic rendering of these sentences. First, to say

> everything is such that it is perfect

is equivalent to saying

> everything has a certain property – it is perfect.

These two sentences are simply verbose ways of saying

> everything is perfect.

Similarly, to say

> at least one thing is such that it is perfect,

which is an alternative to

> there is at least one thing such that it is perfect,

is equivalent to saying

> at least one thing has a certain property – it is perfect.

These two sentences are simply verbose ways of saying

> at least one thing is perfect.

The latter sentence, in turn, can be thought of as one way of rendering precise the following.

> something is perfect

Along similar lines, recall the way that the negation operator works; the official form of negation involves prefixing 'it is not true that' in front of the sentence in question. Thus, for example, one obtains the following.

> it is not true that it is perfect

Recall that this is equivalent to the following more colloquial expression.

> it is not perfect

The advantage of the verbose forms of negation and quantification is grammatical generality; we can always produce the official negation or quantification of a sentence, but we cannot always easily produce the colloquial negation or quantification.

For example, consider the following.

> everything is such that
>   it is not true that
>   it is perfect,

which is equivalent to

> everything is such that
> it is not perfect.

Following the above line of reasoning concerning colloquial quantification, the natural paraphrase of this is the following.

> everything is not perfect

Unfortunately, the placement of ‘not’ in this sentence makes it unclear whether it modifies ‘is’ or ‘perfect’; accordingly, this sentence is ambiguous in meaning between the following pair of sentences.

> everything isn't perfect
> (i.e., not everything is perfect)

> everything is non-perfect

These are not equivalent; if, some things are perfect and some things are not, the first is true, but the second is false.

The original sentence,

> everything is such that it is not perfect,

says that everything has the property of being non-perfect (imperfect), or

> everything is non-perfect (imperfect).

To say that everything is non-perfect (imperfect) is equivalent to saying

> nothing is perfect,

which is much stronger than

> not everything is perfect.

The latter sentence is a colloquial paraphrase of

> it is not true that everything is perfect,

which is a colloquial paraphrase of

> it is not true that
> everything is such that
> it is perfect.

This is precisely formula (4) above.

Now, if not everything is perfect, then there is at least one thing that isn't perfect, and conversely. To say the latter, we write

> at least one thing is such that it is not perfect,

which is formula (8) above.

Finally, consider formula (5)

$\sim\exists xPx$ it is not true that
at least one thing is such that
it is perfect

which is equivalent to

it is not true that
at least one thing is perfect.

The number of things that are perfect is either zero, one, two, three, etc. To say that at least one thing is perfect is to say that the number of perfect things is at least one, that is, the number is not zero. To say that this is not true is to say that the number of perfect things is zero, which is to say

nothing is perfect.

Thus, we basically have six colloquial sentences.

(c1) everything is perfect
(c2) something is perfect (i.e., at least one thing is perfect)
(c3) everything is imperfect

(c4) something is imperfect
(c5) not everything is perfect
(c6) nothing is perfect

These correspond to the following formulas of predicate logic.

(f1) $\forall xPx$
(f2) $\exists xPx$
(f3) $\forall x\sim Px$
(f4) $\exists x\sim Px$
(f5) $\sim\forall xPx$
(f6) $\sim\exists xPx$

As noted earlier, two pairs of formulas are equivalent. In particular:

$\forall x\sim Px$ [everything is imperfect]

is equivalent to

$\sim\exists xPx$ [nothing is perfect],

and

$\exists x\sim Px$ [something is imperfect]

is equivalent to

$\sim\forall xPx$ [not everything is perfect].

These are instances of two very general equivalences, which may be stated as follows.

$\sim\forall x \;=\; \exists x\sim$
$\sim\exists x \;=\; \forall x\sim$

What this means is that for any formula $\mathcal{A}$, however complex, we have the following.

$\sim\forall x\mathcal{A}$ is equivalent to $\exists x\sim\mathcal{A}$.

$\sim\exists x\mathcal{A}$ is equivalent to $\forall x\sim\mathcal{A}$.

In order to understand them better, it might be worthwhile to compare these two equivalences with their counterparts in sentential logic – deMorgan's laws. In their simplest form, these laws of logic are stated as follows.

(dM1) $\sim(\mathcal{A}\&\mathcal{B})$ is equivalent to $\sim\mathcal{A}\vee\sim\mathcal{B}$.
(dM2) $\sim(\mathcal{A}\vee\mathcal{B})$ is equivalent to $\sim\mathcal{A}\&\sim\mathcal{B}$.

But there are more general forms as well, given as follows.

(M1) $\sim(\mathcal{A}_1\&\mathcal{A}_2\&..\&\mathcal{A}_n)$ is equivalent to $\sim\mathcal{A}_1\vee\sim\mathcal{A}_2\vee...\vee\sim\mathcal{A}_n$
(M2) $\sim(\mathcal{A}_1\vee\mathcal{A}_2\vee...\vee\mathcal{A}_n)$ is equivalent to $\sim\mathcal{A}_1\&\sim\mathcal{A}_2\&...\&\sim\mathcal{A}_n$

In other words, the negation of any conjunction, however long, is equivalent to a corresponding disjunction of negations, and similarly, the negation of any disjunction, however long, is equivalent to a corresponding conjunction of negations.

But what does this have to do with universal and existential quantifiers. Well, imagine for a moment there are exactly two things in the universe – call them a and b, respectively. In such a universe, which is very small, every universally quantified statement is equivalent to a conjunction, and every existentially quantified statement is equivalent to a disjunction. In particular, we have the following.

everything is F :: a is F, and b is F

something is F :: a is F, and/or b is F

Or, in formulas:

$\forall xFx$ :: $Fa \& Fb$

$\exists xFx$ :: $Fa \vee Fb$

Similarly, if there are exactly three things in the universe (a, b, c), then we have the following equivalences.

everything is F :: a is F, and b is F, and c is F

something is F :: a is F, and/or b is F, and/or c is F

Or, in formulas:

$\forall x Fx$ :: $Fa \,\&\, Fb \,\&\, Fc$

$\exists x Fx$ :: $Fa \vee Fb \vee Fc$

This can be generalized to any (finite) number of things in the universe; for every universally/ existentially quantified statement, there is a corresponding conjunction/ disjunction of suitable length.

Having seen what the equivalence looks like in general, let us concentrate on the simplest non-trivial version – a universe with just two things (a and b) in it.

Next, let us consider what happens when we combine quantifiers with negation? First, the simplest.

everything is not-F :: a is not F and b is not F

something is not-F :: a is not F and/or b is not F

Or, in formulas:

$\forall x{\sim}Fx$ :: ${\sim}Fa \,\&\, {\sim}Fb$

$\exists x{\sim}Fx$ :: ${\sim}Fa \vee {\sim}Fb$

Negating the quantified statements yields:

not everything is F :: not(a is F and b is F)

nothing is F :: not something is F :: not(a is F and/or b is F)

Or, in formulas:

${\sim}\forall x Fx$ :: ${\sim}(Fa \,\&\, Fb)$

${\sim}\exists x Fx$ :: ${\sim}(Fa \vee Fb)$

Finally, we obtain the following chain of equivalences.

${\sim}\forall x Fx$ :: ${\sim}(Fa \,\&\, Fb)$ :: ${\sim}Fa \vee {\sim}Fb$ :: $\exists x{\sim}Fx$

${\sim}\exists x Fx$ :: ${\sim}(Fa \vee Fb)$ :: ${\sim}Fa \,\&\, {\sim}Fb$ :: $\forall x{\sim}Fx$

The same procedure can be carried out with three, or four, or any number of, individuals.

Note: In the previous example, the formula $\mathcal{A}$ is simple, being Fx. In general, $\mathcal{A}$ may be complex – for example, it might be the formula $(Fx \to Gx)$. Then ${\sim}\mathcal{A}$ is the negation of the entire formula, which is ${\sim}(Fx \to Gx)$. (Notice that the parentheses are optional in the conditional, but not in its negation.)

10. SYMBOLIZING THE STATEMENT FORMS OF SYLLOGISTIC LOGIC

Recall that the statement forms of syllogistic logic are given as follows.

(f1) all A are B
(f2) some A are B
(f3) no A are B
(f4) some A are not B

These are all stated in the plural form. In order to translate these into predicate logic, the first thing we must do is to convert each plural form into the corresponding closest singular form.

(s1)	every A is B	[every A is a B]
(s2)	some A is B	[some A is a B]
(s3)	no A is B	[no A is a B]
(s4)	some A is not B	[some A is not a B]

Examples of sentences in these forms are given as follows.

(e1) every astronaut is brave
(e2) some astronaut is brave
(e3) no astronaut is brave
(e4) some astronaut is not brave

Note that the simple predicate 'is brave' can be replaced by the longer expression 'is a brave person'.

The next thing we must do is to convert the specific quantifier expressions 'every/some/no A' into the corresponding expressions involving general quantifiers 'every/some/thing is such that...'

Consider (s1); to say

every A is a B

is to say

everything that is A is B,

or if we have persons exclusively in mind,

everyone who is A is B.

For example, we could read the latter as follows.

everyone who is an astronaut is brave

We know how to formalize 'everything (everyone) is B'.

everything is such that it is B $\forall xBx$

But we don't want to say that everything is B, just every A is B. How do we add the clause 'that (who) is A'? Let us try the following paraphrases.

> everything is B *provided* it is A
>
> everything is such that it is B *provided* it is A

Now we are getting somewhere, since this sentence divides as follows.

> everything is such that
> it is B provided it is A

Adding the crucial pronoun indices (variables), we obtain the following.

> everything x is such that
> x is B provided x is A

Recall '$\mathcal{B}$ provided $\mathcal{A}$' is equivalent to '$\mathcal{B}$ if $\mathcal{A}$', which is equivalent to 'if $\mathcal{A}$, then $\mathcal{B}$', which is symbolized $\mathcal{A}\rightarrow\mathcal{B}$. Thus, the above sentence is symbolized as follows:

$$\forall x(Ax \rightarrow Bx).$$

Note carefully the parentheses around the conditional; it's OK to omit them when the formula stands by itself, but when it goes into making a larger formula, the outer parentheses must be restored. The same thing happens when we negate a conditional.

Of course, the corresponding formula without parentheses,

$$\forall xAx \rightarrow Bx,$$

is also a formula of predicate logic, just as $\sim A \rightarrow B$ is a formula of sentential logic. Both are conditionals. The latter says 'if not A, then B', in contrast to 'it is not true that if A then B', which is the reading of $\sim(A \rightarrow B)$. The most accurate translation of the predicate logic formula, which is logically equivalent to

$$\forall xAx \rightarrow By,$$

reads as follows.

> if everything is A, then *this* is B,

where 'this' points at something external to the sentence. This is a perfectly good piece of English, but it is definitely not the same as saying that every A is B.

Next, let us consider (s2) above. To say

> some A is B,

for example, to say

> some astronaut is brave,

is to say

there is at least one A that (who) is also B,

which is equivalent to

there is at least one A *and* it (he/she) is also B.

Notice that the pronoun ‘it’ points internally at ‘at least one A’.

We know how to say

there is at least one A.
there is at least one thing such that it is A
∃xAx

How do we add the clause ‘that is also B’ or ‘and it is also B’? Well, we are saying that the thing in question is A, and we are saying *in addition* that it is B, so we are saying that it is A *and* it is B, which gives us the following.

there is at least one thing such that
  it is A
    and it is B

This is symbolized as follows.

∃x(Ax & Bx)

Notice once again that the outer parentheses are restored before the quantifier is prefixed. If we were to drop the parentheses, we obtain

∃xAx & Bx,

which is logically equivalent to

∃xAx & By,

which may be read

something is A, and *this* is B,

where ‘this’ points externally at whatever the person using this sentence is pointing toward. Although this is a perfectly good formula of predicate logic, it says something entirely different from ‘some A is B’

Next, let us consider (s3) above. To say

no A is B,

for example,

no astronaut is brave,

is to *deny* that there is at least one A who is B. In other words, it is the negation of ‘some A is B’, and is accordingly symbolized as follows,

∼∃x(Ax & Bx),

which is literally read as

> it is not true that
> > there is at least one thing such that
> > > it is A and it is B

Recall that $\sim\exists x\mathbb{F}$ is equivalent to $\forall x\sim\mathbb{F}$, for any formula $\mathbb{F}$. In the above case, $\mathbb{F}$ is the formula (Ax&Bx), we have the following equivalence.

$$\sim\exists x(Ax \ \&\ Bx) \ :: \ \forall x\sim(Ax \ \&\ Bx)$$

But, in sentential logic, we have the following equivalence (check the truth table!)

$$\sim(\mathcal{A} \ \&\ \mathcal{B}) \ :: \ \mathcal{A} \rightarrow \sim\mathcal{B}$$

So, putting these together, we obtain the following equivalence.

$$\sim\exists x(Ax \ \&\ Bx) \ :: \ \forall x(Ax \rightarrow \sim Bx)$$

Thus, we have an alternative way of formulating ‘no A is B’:

$$\forall x(Ax \rightarrow \sim Bx),$$

which is read literally as

> everything is such that
> > if it is A
> > > then it is not B

Finally, let us consider (s4) above. To say

> some A is not B

is to say

> there is at least one A and it is not B,

which is symbolized very much the same way as ‘some A is B’,

$$\exists x(Ax \ \&\ \sim Bx),$$

which is read literally as follows.

> there is at least one thing such that
> > it is A and it is not B

Let us compare this with the following negation,

> not every A is B,

which is symbolized just like

> it is not true that every A is B,

thus:

$$\sim\forall x(Ax \rightarrow Bx),$$

whose literal reading is

it is not true that
every thing is such that
if it is A then it is B.

Recall that $\sim\forall x\mathbb{F}$ is equivalent to $\exists x\sim\mathbb{F}$, for any formula $\mathbb{F}$; in the above case $\mathbb{F}$ is the formula $(Ax \rightarrow Bx)$ – notice the parentheses – so we obtain the following equivalence.

$$\sim\forall x(Ax \rightarrow Bx) \ :: \ \exists x\sim(Ax \rightarrow Bx)$$

But recall the following equivalence of sentential logic.

$$\sim(\mathcal{A} \rightarrow \mathcal{B}) \ :: \ \mathcal{A} \ \& \sim\mathcal{B}$$

Thus, we have the following equivalence of predicate logic.

$$\sim\forall x(Ax \rightarrow Bx) \ :: \ \exists x(Ax \ \& \sim Bx)$$

In other words, to say

not every A is B

is the same as to say

some A is not B.

For example, the following in effect say the same thing.

not every astronaut is brave
some astronaut is not brave

11. SUMMARY OF THE BASIC QUANTIFIER TRANSLATION PATTERNS SO FAR EXAMINED

Before continuing, it is a good idea to review the basic patterns of translation that we have examined so far. These are given as follows.

Simple Quantification Plus Negation

(1)	everything is B	$\forall x Bx$
(2)	something is B	$\exists x Bx$
(3)	nothing is B	$\sim\exists x Bx$
(4)	something is non-B	$\exists x \sim Bx$
(5)	everything is non-B	$\forall x \sim Bx$
(6)	not everything is B	$\sim\forall x Bx$

Syllogistic Forms Plus Negation

(7)	every A is B	$\forall x(Ax \rightarrow Bx)$
(8)	some A is B	$\exists x(Ax \& Bx)$
(9)	no A is B	$\sim\exists x(Ax \& Bx)$
(10)	some A is not B	$\exists x(Ax \& \sim Bx)$
(11)	every A is a non-B	$\forall x(Ax \rightarrow \sim Bx)$
(12)	not every A is B	$\sim\forall x(Ax \rightarrow Bx)$

In addition to these, it is important to keep the following logical equivalences in mind when doing translations into predicate logic.

Basic Logical Equivalences

(1)	$\sim\exists x Ax$	::	$\forall x \sim Ax$
(2)	$\sim\forall x Ax$	::	$\exists x \sim Ax$
(3)	$\sim\exists x(Ax \& Bx)$	::	$\forall x(Ax \rightarrow \sim Bx)$
(4)	$\sim\forall x(Ax \rightarrow Bx)$	::	$\exists x(Ax \& \sim Bx)$

In looking over the above patterns, one might wonder why the following is not a correct translation:

(1) every A is B $\forall x(Ax \& Bx)$ WRONG!!!

The correct translation is given as follows.

(2) every A is B $\forall x(Ax \rightarrow Bx)$ RIGHT!!!

Remember there simply is no general symbol-by-symbol translation between colloquial English and the language of predicate logic; in the correct translation (2), no symbol in the formula corresponds to the 'is' in the colloquial sentence, and no symbol in the colloquial English sentence corresponds to '→' in the formula.

The erroneous nature of (1) becomes apparent as soon as we translate the formula into English, which goes as follows.

everything is such that
  it is A *and* it is B

For example,

everything is such that
  it is an astronaut *and* it is brave

In other words,

everything is an astronaut who is brave,

or equivalently,

everything is a brave astronaut.

This is also equivalent to:

everything is an astronaut *and* everything is brave.

Needless to say, this does *not* say the same thing as:

every astronaut is brave.

O.K., arrow works when we have 'every A is B', but ampersand does not work. So, why doesn't arrow work just as well in the corresponding statement 'some A is B'? Why isn't the following a correct translation?

(3) some A is B $\exists x(Ax \rightarrow Bx)$ WRONG!!!

As noted above, the correct translation is:

(4) some A is B $\exists x(Ax \,\&\, Bx)$ RIGHT!!!

Once again, please note that there is no symbol-by-symbol translation between the colloquial English form and the predicate logic formula.

Let's see what happens when we translate the formula of (3) into English; the straight translation yields the following:

(3t) there is at least one thing such that
  if it is A **then** it is B.

Does this say that some A is B? No! In fact, it is not clear what it says. If the conditional *were* subjunctive, rather than truth-functional, then (3t) might correspond to the following colloquial subjunctive sentence.

there is someone who
would be brave if he *were* an astronaut

From this, it surely does not follow that there is even a single brave astronaut, or even a single astronaut. To make this clear, consider the following analogous sentence.

some Antarctican is brave

Here, let us understand 'Antarctican' to mean a permanent citizen of Antarctica. This sentence must be carefully distinguished from the following.

there is someone who
would be brave if he/she were Antarctican

To say that some Antarctican is brave to say that there is at least one Antarctican who is brave, from which it obviously follows that there is at least one Antarctican. The sentence 'some Antarctican is brave' logically implies 'at least one Antarctican exists'.

By contrast, the sentence 'there is someone who would be brave if *he/she* were Antarctican' does not imply that any Antarctican exists. Whether there is such a person who would be brave were he/she to become an Antarctican, I really couldn't say, but I suspect it is probably true. It takes a brave person to live in Antarctica.

When we take if-then as a subjunctive conditional, we see very quickly that $\exists x(Ax \rightarrow Bx)$ simply does not say that some A is B. What happens if we insist that if-then is truth-functional? In that case, the sentence $\exists x(Ax \rightarrow Bx)$ is automatically true, so long as we can find someone who is *not* Antarctican!

Suppose that Smith is *not* Antarctican. Then the sentence

Smith is Antarctican

is false, and hence the conditional sentence

if Smith is Antarctican, then Smith is brave

is true! Why? Because of the truth table for if-then! But if Smith is such that if he is Antarctican then he is brave, then at least one person is such that if he is Antarctican then he is brave. Thus, the following existential sentence is true.

there is someone such that
if he is Antarctican,
then he is brave

We conclude this section by presenting the following rule of thumb about how symbolizations *usually* go. Of course, in saying that it is a rule of thumb, all one means is that it works quite often, not that it works always.

Rule of Thumb (not absolute)

If one has a universal formula,
then the connective immediately "beneath"
the universal quantifier
is a conditional.

If one has an existential formula,
then the connective immediately "beneath"
the existential quantifier
is a conjunction.

The slogan that goes with this reads as follows:

UNIVERSAL-CONDITIONAL

EXISTENTIAL-CONJUNCTION

Remember! This is just a rule of thumb! There are numerous exceptions, which will be presented in subsequent sections.

12. FURTHER TRANSLATIONS INVOLVING SINGLE QUANTIFIERS

In the previous section, we saw how one can formulate the statement forms of syllogistic logic in terms of predicate logic. However, the expressive power of predicate logic is significantly greater than syllogistic logic. Syllogistic patterns are a very tiny fraction of the statement forms that can be formulated in predicate logic.

In the next three sections (Sections 12-14), we are going to explore numerous patterns of predicate logic that all have one thing in common with what we have so far examined. Specifically, they all involve exactly one quantifier. More specifically still, each one has one of the following forms.

(f1) $\forall x\mathcal{A}$
(f2) $\exists x\mathcal{A}$
(f3) $\sim\forall x\mathcal{A}$
(f4) $\sim\exists x\mathcal{A}$

In particular, either the main connective is a quantifier, or the main connective is negation, and the next connective is a quantifier.

We have already seen the simplest examples of these forms, in Sections 10 and 11.

$\exists x Bx$	something is B
$\forall x Bx$	everything is B
$\sim\exists x Bx$	nothing is B
$\sim\forall x Bx$	not everything is B
$\exists x \sim Bx$	something is non-B
$\forall x \sim Bx$	everything is non-B

We can also formulate sentences that have an overall form like one of the above, but which have more complicated formulas in place of 'Bx'. The following are examples.

(1) everything is both A and B
(2) everything is either A or B
(3) everything is A but not B
(4) something is both A and B
(5) something is either A or B
(6) something is A but not B
(7) nothing is both A and B
(8) nothing is either A or B
(9) nothing is A but not B

How do we translate these sorts of sentences into predicate logic? One way is first to notice that the *overall* forms of these sentences may be written and symbolized, respectively, as follows.

(o1)	everything is J	$\forall x Jx$
(o2)	everything is K	$\forall x Kx$
(o2)	everything is L	$\forall x Lx$
(o3)	something is J	$\exists x Jx$
(o4)	something is K	$\exists x Kx$
(o2)	something is L	$\exists x Lx$
(o5)	nothing is J	$\sim\exists x Jx$
(o6)	nothing is K	$\sim\exists x Kx$
(o2)	nothing is L	$\sim\exists x Lx$

Here, the pseudo-atomic formulas Jx, Kx, and Lx are respectively short for the more complex formulas, given as follows.

Jx	::	$(Ax \,\&\, Bx)$
Kx	::	$(Ax \vee Bx)$
Lx	::	$(Ax \,\&\, \sim Bx)$

Note the appearance of the outer parentheses. Substituting in accordance with these equivalences, we obtain the following translations of the above sentences.

(t1) $\forall x(Ax \& Bx)$
(t2) $\forall x(Ax \vee Bx)$
(t3) $\forall x(Ax \& \sim Bx)$
(t4) $\exists x(Ax \& Bx)$
(t5) $\exists x(Ax \vee Bx)$
(t6) $\exists x(Ax \& \sim Bx)$
(t7) $\sim\exists x(Ax \& Bx)$
(t8) $\sim\exists x(Ax \vee Bx)$
(t9) $\sim\exists x(Ax \& \sim Bx)$

The following paraphrase chains may help to see how one might go about producing the symbolization.

(c1) everything is both A and B

everything is such that
it is both A and B

everything is such that
it is A and it is B

$\forall x(Ax \& Bx)$

(c2) everything is either A or B

everything is such that
it is either A or B

everything is such that
it is A or it is B

$\forall x(Ax \vee Bx)$

(c3) everything is A but not B

everything is such that
it is A but not B

everything is such that
it is A and it is not B

$\forall x(Ax \& \sim Bx)$

(c4) something is both A and B

there is at least one thing such that
it is both A and B

there is at least one thing such that
it is A and it is B

$\exists x(Ax \& Bx)$

You will recall, of course, that 'something is both A and B' is logically equivalent to 'some A is B', as noted in the previous sections.

(c5) something is either A or B

there is at least one thing such that
it is either A or B

there is at least one thing such that
it is A or it is B

$\exists x(Ax \vee Bx)$

(c6) something is A but not B

there is at least one thing such that
it is A but not B

there is at least one thing such that
it is A and it is not B

$\exists x(Ax \& \sim Bx)$

(c7) nothing is both A and B

it is not true that something is both A and B

it is not true that
there is at least one thing such that
it is both A and B

it is not true that
there is at least one thing such that
it is A and it is B

$\sim\exists x(Ax \& Bx)$

(c8) nothing is either A or B

it is not true that something is either A or B

it is not true that there is at least one thing such that
it is either A or B

it is not true that there is at least one thing such that
it is A or it is B

$\sim\exists x(Ax \vee Bx)$

(c9) nothing is A but not B

it is not true that something A but not B

it is not true that there is at least one thing such that
it is A but not B

it is not true that there is at least one thing such that
it is A and it is not B

$\sim\exists x(Ax \,\&\, \sim Bx)$

In the next section, we will further examine these kinds of sentences, but will introduce a further complication.

13. CONJUNCTIVE COMBINATIONS OF PREDICATES

So far, we have concentrated on formulas that have at most two predicates. In the present section, we drop that restriction and discuss formulas with three or more predicates. However, for the most part, we will concentrate on conjunctive combinations of predicates.

Consider the following sentences (which pertain to a fictional group of people, called Bozonians, who inhabit the fictional country of Bozonia).

E:
(e1) every Adult Bozonian is a Criminal
(e2) every Adult Criminal is a Bozonian
(e3) every Criminal Bozonian is an Adult
(e4) every Adult is a Criminal Bozonian
(e5) every Bozonian is an Adult Criminal
(e6) every Criminal is an Adult Bozonian

S:
(s1) some Adult Bozonian is a Criminal
(s2) some Adult Criminal is a Bozonian
(s3) some Criminal Bozonian is an Adult
(s4) some Adult is a Criminal Bozonian
(s5) some Bozonian is an Adult Criminal
(s6) some Criminal is an Adult Bozonian

N:
(n1) no Adult Bozonian is a Criminal
(n2) no Adult Criminal is a Bozonian
(n3) no Criminal Bozonian is an Adult
(n4) no Adult is a Criminal Bozonian
(n5) no Bozonian is an Adult Criminal
(n6) no Criminal is an Adult Bozonian

The predicate terms have been capitalized for easy spotting. The official predicates are as follows.

A: ...is an adult
B: ...is a Bozonian
C: ...is a criminal

You will notice that every sentence above involves at least one of the following predicate combinations.

AB: adult Bozonian (Bozonian adult)
AC: adult criminal (criminal adult)
BC: Bozonian criminal (criminal Bozonian)

In these particular cases, the predicates combine in the simplest manner possible – i.e., *conjunctively*. In other words, the following are equivalences for the complex predicates.

x is an Adult Bozonian	::	x is an Adult and x is a Bozonian
x is an Adult Criminal	::	x is an Adult and x is a Criminal
x is a Bozonian Criminal	::	x is a Bozonian and x is a Criminal

The above predicates combine *conjunctively*; this is *not* a universal feature of English, as evidenced by the following examples.

x is an alleged criminal
x is a putative solution
x is imitation leather
x is an expectant mother
x is an experienced sailor; x is an experienced hunter
x is a large whale; x is a small whale
x is a large shrimp; x is a small shrimp
x is a deer hunter; x is a shrimp fisherman

For example, an alleged criminal is not a criminal who is alleged; indeed an alleged criminal need not be a criminal at all. Similarly, an expectant mother need not be a mother at all.

By contrast, an experienced sailor is a sailor, but not a sailor who is generally experienced. Similarly, an experienced hunter is a hunter, but not a hunter who is generally experienced. In each case, the person is not experienced in general, but rather is experienced *at* a particular thing (sailing, hunting).

Along the same lines, a large whale is a whale, and a large shrimp is a shrimp, but neither is generally large; neither is nearly as large as a small ocean, let alone a small planet, or a small galaxy.

Finally a deer hunter is not a deer who hunts, but someone or something that hunts deer, and a shrimp fisherman is not a shrimp who fishes but someone who fishes for shrimp.

I am sure that the reader can come up with numerous other examples of predicates that don't combine conjunctively.

Sometimes, a predicate combination is ambiguous between a conjunctive and a non-conjunctive reading. The following is an example.

x is a Bostonian Cabdriver

This has a conjunctive reading.

x is a Bostonian who drives a cab
(perhaps in Boston, perhaps elsewhere)

But it also has a non-conjunctive reading.

x is a person who drives a cab in Boston
(who lives perhaps in Boston, perhaps elsewhere)

Another example, which seems to engender confusion is the following.

x is a male chauvinist

This has a conjunctive reading,

x is a male and x is a chauvinist,

which means

x is a male who is excessively (and blindly) patriotic (loyal).

However, this is not what is usually meant by the phrase ‘male chauvinist’. As originally intended by the author of this phrase, a male chauvinist need not be male, and a male chauvinist need not be a chauvinist. Rather, a male chauvinist is a person (male or female) who is excessively (and blindly) loyal *in respect to* the alleged superiority of men to women.

It is important to realize that many predicates don't combine conjunctively. Nonetheless, we are going to concentrate exclusively on ones that do, for the sake of simplicity. When there are two readings of a predicate combination, we will opt for the conjunctive reading, and ignore the non-conjunctive reading.

Now, let's go back to the original problem of paraphrasing the various sentences concerning adults, Bozonians, and criminals. We do two examples from each group, in each case by presenting a paraphrase chain.

(e1) every Adult Bozonian is a Criminal

every AB is C

everything is such that

if it is AB, then it is C

everything is such that
if it is A and it is B, then it is C

$\forall x([Ax \;\&\; Bx] \rightarrow Cx)$

(e4) every Adult is a Bozonian Criminal

every A is BC

everything is such that
if it is A, then it is BC

everything is such that
if it is A, then it is B and it is C

$\forall x(Ax \rightarrow [Bx \;\&\; Cx])$

(s3) some Criminal Bozonian is an Adult

some CB is A

there is at least one thing such that
it is CB, and it is A

there is at least one thing such that
it is C and it is B, and it is A

$\exists x([Cx \;\&\; Bx] \;\&\; Ax)$

(s5) some Bozonian is an Adult Criminal

some B is AC

there is at least one thing such that
 it is B, and it is AC

there is at least one thing such that
 it is B, and it is A and it is C

∃x(Bx & [Ax & Cx]).

(n3) no Criminal Bozonian is an Adult

no CB is A

it is not true that some CB is A

it is not true that
 there is at least one thing such that
  it is CB, and it is A

 it is not true that
  there is at least one thing such that
   it is C and it is B, and it is A

~∃x([Cx & Bx] & Ax).

(n6) no Criminal is an Adult Bozonian

no C is AB

it is not true that some C is AB

it is not true that
 there is at least one thing such that
  it is C, and it is AB

it is not true that
 there is at least one thing such that
  it is C, and it is A and it is B

~∃x(Cx & [Ax & Bx]).

The reader is invited to symbolize the remaining sentences from the above groups.

We can further complicate matters by adding an additional predicate letter (say) 'D', which symbolizes (say) '___ is deranged'. Consider the following two examples.

(e1) every Deranged Adult is a Criminal Bozonian
(e2) no Adult Bozonian is a Deranged Criminal

The symbolizations go as follows.

(s1) $\forall x([Dx \,\&\, Ax] \rightarrow [Cx \,\&\, Bx])$
(s2) $\sim\exists x([Ax \,\&\, Bx] \,\&\, [Dx \,\&\, Cx])$

Another possible complication concerns internal negations in the sentences. The following are examples, together with their step-wise paraphrases.

(1) every Adult who is not Bozonian is a Criminal

every A who is not B is C

everything is such that
if it is an A who is not B,
then it is C

everything is such that
if it is A and it is not B,
then it is C

$\forall x([Ax \,\&\, \sim Bx] \rightarrow Cx)$

(2) some Adult Bozonian is not a Criminal

some AB is not C

there is at least one thing such that
it is AB, and it is not C

there is at least one thing such that
it is A and it is B, and it is not C

$\exists x([Ax \,\&\, Bx] \,\&\, \sim Cx)$

(3) some Bozonian is an Adult who is not a Criminal

some B is an A who is not a C

there is at least one thing such that
it is B, and it is an A who is not C

there is at least one thing such that
it is B, and it is A and it is not C

$\exists x(Bx \,\&\, [Ax \,\&\, \sim Cx])$

(4) no Adult who is not a Bozonian is a Criminal

no A who is not B is C

it is not true that
    some A who is not B is C

it is not true that
    there is at least one thing such that
        it is an A who is not B, and it is C

it is not true that
    there is at least one thing such that
        it is A and it is not B, and it is C

$\sim\exists x([Ax \mathbin{\&} \sim Bx] \mathbin{\&} Cx)$

14. SUMMARY OF BASIC TRANSLATION PATTERNS FROM SECTIONS 12 AND 13

Forms With Only Two Predicates

(1)	everything is both A and B	$\forall x(Ax \& Bx)$
(2)	everything is A but not B	$\forall x(Ax \& {\sim}Bx)$
(3)	everything is either A or B	$\forall x(Ax \vee Bx)$

(1)	something is both A and B	$\exists x(Ax \& Bx)$
(2)	something is A but not B	$\exists x(Ax \& {\sim}Bx)$
(3)	something is either A or B	$\exists x(Ax \vee Bx)$

(1)	nothing is both A and B	${\sim}\exists x(Ax \& Bx)$
(2)	nothing is A but not B	${\sim}\exists x(Ax \& {\sim}Bx)$
(3)	nothing is either A or B	${\sim}\exists x(Ax \vee Bx)$

Simple Conjunctive Combinations

(1)	every AB is C	$\forall x([Ax \& Bx] \rightarrow Cx)$
(2)	some AB is C	$\exists x([Ax \& Bx] \& Cx)$
(3)	some AB is not C	$\exists x([Ax \& Bx] \& {\sim}Cx)$
(4)	no AB is C	${\sim}\exists x([Ax \& Bx] \& Cx)$

(5)	every A is BC	$\forall x(Ax \rightarrow [Bx \& Cx])$
(6)	some A is BC	$\exists x(Ax \& [Bx \& Cx])$
(7)	some A is not BC	$\exists x(Ax \& {\sim}[Bx \& Cx])$
(8)	no A is BC	${\sim}\exists x(Ax \& [Bx \& Cx])$

Conjunctive Combinations Involving Negations

(1)	every A that is not B is C	$\forall x([Ax \& {\sim}Bx] \rightarrow Cx)$
(2)	some A that is not B is C	$\exists x([Ax \& {\sim}Bx] \& Cx)$
(3)	some A that is not B is not C	$\exists x([Ax \& {\sim}Bx] \& {\sim}Cx)$
(4)	no A that is not B is C	${\sim}\exists x([Ax \& {\sim}Bx] \& Cx)$

(5)	every A is B but not C	$\forall x(Ax \rightarrow [Bx \& {\sim}Cx])$
(6)	some A is B but not C	$\exists x(Ax \& [Bx \& {\sim}Cx])$
(7)	no A is B but not C	${\sim}\exists x(Ax \& [Bx \& {\sim}Cx])$

15. 'ONLY'

The standard quantifiers of predicate logic are 'every' and 'at least one'. We have already seen how to paraphrase various non-standard quantifiers into standard form. In particular, we paraphrase 'all' as 'every', 'some' as 'at least one', and 'no' as 'not at least one'.

In the present section, we examine another non-standard quantifier, 'only'; in particular, we show how it can be paraphrased using the standard quantifiers. In a later section, we examine a subtle variant – 'the only'. But for the moment let us concentrate on 'only' by itself.

The basic quantificational form for 'only' is:

only $\mathbb{A}$ are $\mathbb{B}$.

Examples include:

(1) only Men are NFL football players
(2) only Citizens are Voters

Occasionally, signs use 'only' as in:

employees only
members only
passenger cars only

These can often be paraphrased as follows.

(3) only Employees are Allowed
(4) only Members are Allowed
(5) only Passenger cars are Allowed

What is, in fact, allowed (or disallowed) depends on the context. Generally, signs employing 'only' are intended to *exclude* certain things, specifically things that *fail* to have a certain property (being an employee, being a member, being a passenger car, etc.).

Before dealing with the quantifier 'only', let us recall a similar expression in sentential logic – namely, 'only if'. In particular, recall that

A only if B

may be paraphrased as

not A if *not* B,

which in standard form is written

if not B, then not A [$\sim B \rightarrow \sim A$]

In other words, 'only' modifies 'if' by introducing two negations. The word 'if' always introduces the antecedent, and the word 'only' modifies 'if' by adding two negations in the appropriate places.

When combined with the connective 'if', the word 'only' behaves as a special sort of double-negative modifier. When 'only' acts as a quantifier, it behaves in a similar, double-negative, manner. Recall the signs involving 'only'; they are intended to *exclude* persons who *fail* to have a certain property.

Indeed, we can paraphrase 'only $\mathcal{A}$ are $\mathcal{B}$' in at least two very different ways involving double-negatives.

First, we can paraphrase 'only $\mathcal{A}$ are $\mathcal{B}$' using the negative quantifier 'no', as follows

(o) only $\mathbb{A}$ are $\mathbb{B}$

(p) *no non* $\mathbb{A}$ are $\mathbb{B}$

Strictly speaking, 'non' is not an English word, but simply a prefix; properly speaking, we should write the following.

(p*) *no non*-$\mathbb{A}$ are $\mathbb{B}$

However, the hyphen will generally be dropped, simply to avoid clutter in our intermediate symbolizations.

Thus, the following is the "skeletal" paraphrase:

only = no non [only = no non-]

However, in various colloquial examples, the following more "meaty" paraphrase is more suitable.

(p) *no one* who is *not* $\mathbb{A}$ is $\mathbb{B}$

So, for example, (1)-(4) may be paraphrased as follows.

(p1) no one who isn't a Man is an NFL football player
(p2) no one who isn't a Citizen is a Voter
(p3) no one who isn't an Employee is Allowed
(p4) no one who isn't a Member is Allowed

Next, we turn to symbolization. First the general form is:

(o) only $\mathbb{A}$ are $\mathbb{B}$,

which is paraphrased:

(p) no non $\mathbb{A}$ are $\mathbb{B}$ [no non $\mathbb{A}$ is $\mathbb{B}$]

This is symbolized as follows.

(s) $\sim\exists x(\sim\mathbb{A}x \,\&\, \mathbb{B}x)$

Similarly, (o1)-(o5) are symbolized as follows.

(s1) $\sim\exists x(\sim Mx \,\&\, Nx)$
(s2) $\sim\exists x(\sim Cx \,\&\, Vx)$
(s3) $\sim\exists x(\sim Ex \,\&\, Ax)$
(s4) $\sim\exists x(\sim Mx \,\&\, Ax)$
(s5) $\sim\exists x(\sim Px \,\&\, Ax)$

The quickest way to paraphrase ‘only’ is using the equivalence

ONLY = NO NON

An alternative paraphrase technique uses ‘all’/‘every’ plus two occurrences of ‘non’/‘not’, as follows

(o) only $\mathbb{A}$ are $\mathbb{B}$
(p1) all non $\mathbb{A}$ are non $\mathbb{B}$
(p2) every non $\mathbb{A}$ is non $\mathbb{B}$
(p3) everyone who is *not* $\mathbb{A}$ is *not* $\mathbb{B}$

These are symbolized as follows.

(s) $\forall x(\sim\mathbb{A}x \rightarrow \sim\mathbb{B}x)$

So, for example, (1)-(5) may be paraphrased as follows.

(p1) everyone who isn't a Man isn't an NFL football player
(p2) everyone who isn't a Citizen isn't a Voter
(p3) everyone who isn't an Employee isn't Allowed
(p4) everyone who isn't a Member isn't Allowed
(p5) everyone who isn't (driving) a Passenger car isn't Allowed

These in turn are symbolized as follows.

(s1) $\forall x(\sim Mx \rightarrow \sim Nx)$
(s2) $\forall x(\sim Cx \rightarrow \sim Vx)$
(s3) $\forall x(\sim Ex \rightarrow \sim Ax)$
(s4) $\forall x(\sim Mx \rightarrow \sim Ax)$
(s5) $\forall x(\sim Px \rightarrow \sim Ax)$

The two approaches above are equivalent, since the following is an equivalence of predicate logic.

$$\sim\exists x(\sim\mathbb{A}x \,\&\, \mathbb{B}x) \;::\; \forall x(\sim\mathbb{A}x \rightarrow \sim\mathbb{B}x)$$

To see this equivalence, first recall the following quantificational equivalence:

$$\sim\exists x\mathcal{F} \;::\; \forall x{\sim}\mathcal{F}$$

And recall the following sentential equivalence:

$$\sim(\sim\mathcal{A} \,\&\, \mathcal{B}) \;::\; \sim\mathcal{A} \rightarrow \sim\mathcal{B}$$

Accordingly,

$$\sim\exists x(\sim\mathbb{A}x \,\&\, \mathbb{B}x) \;::\; \forall x\sim(\sim\mathbb{A}x \,\&\, \mathbb{B}x)$$

And

$$\sim(\sim\mathbb{A}x \,\&\, \mathbb{B}x) \;::\; (\sim\mathbb{A}x \rightarrow \sim\mathbb{B}x)$$

So

$$\sim\exists x(\sim\mathbb{A}x \,\&\, \mathbb{B}x) \;::\; \forall x(\sim\mathbb{A}x \rightarrow \sim\mathbb{B}x)$$

There is still another sentential equivalence:

$$\sim\mathcal{A} \rightarrow \sim\mathcal{B} \;::\; \mathcal{B} \rightarrow \mathcal{A}$$

So

$$\sim\mathbb{B}x \rightarrow \sim\mathbb{A}x \;::\; (\mathbb{B}x \rightarrow \mathbb{A}x)$$

So

$$\sim\exists x(\sim\mathbb{A}x \,\&\, \mathbb{B}x) \;::\; \forall x(\mathbb{B}x \rightarrow \mathbb{A}x)$$

This equivalence enables us to provide yet another paraphrase and symbolization of ‘only $\mathbb{A}$ are $\mathbb{B}$’, as follows.

(o) only $\mathbb{A}$ are $\mathcal{B}$
(p) all $\mathbb{B}$ are $\mathbb{A}$
(s) $\forall x(\mathbb{B}x \rightarrow \mathbb{A}x)$

The latter symbolization is admitted in intro logic, just as $P\rightarrow Q$ is admitted as a symbolization of ‘P only if Q’, in addition to the official $\sim Q\rightarrow\sim P$. The problem is that the non-negative construals of ‘only’ statements sound funny (even wrong, to some people) in English.

In short, our official paraphrase/symbolization goes as follows.

(o) only $\mathbb{A}$ are $\mathbb{B}$

(p1) *no non* $\mathbb{A}$ is $\mathbb{B}$
(s1) $\sim\exists x(\sim\mathbb{A}x \,\&\, \mathbb{B}x)$

(p2) every *non* $\mathbb{A}$ is *non* $\mathbb{B}$
(s2) $\forall x(\sim\mathbb{A}x \rightarrow \sim\mathbb{B}x)$

Note carefully, however, for the sake of having a single form, the *former* paraphrase/symbolization will be used *exclusively* in the answers to the exercises.

16. AMBIGUITIES INVOLVING 'ONLY'

Having discussed the basic 'only' statement forms, we now move to examples involving more than two predicates. As it turns out, adding a third predicate can complicate matters.

Consider the following example.

(e1) only Poisonous Snakes are Dangerous

Let us assume that 'poisonous' combines conjunctively, so that a poisonous snake is simply a snake that is poisonous, even though a poisonous snake is quite different from a poisonous mushroom (a mushroom's bite is not very deadly!) Granting this simplifying assumption, we have the following paraphrase.

x is a Poisonous Snake :: x is Poisonous and x is a Snake

Now, if we follow the pattern of paraphrase suggested in the previous section, we obtain the following paraphrase.

(p1) no non Poisonous Snakes are Dangerous
no non Poisonous Snake is Dangerous

Unfortunately, the scope of 'non' is ambiguous. For the sentence

(1) x is a non poisonous snake

has two different readings, and hence two different symbolizations.

(r1) x is a non-poisonous snake
(r2) x is a non(poisonous snake)
x is not a poisonous snake

(s1) ~Px & Sx
(s2) ~(Px & Sx)

On one reading, to be a non poisonous snake is to be a snake that is not poisonous. On the other reading, to be a non poisonous snake is simply to be *anything but* a poisonous snake.

Our original sentence, and its paraphrase,

only poisonous snakes are dangerous
no non poisonous snakes are dangerous

are correspondingly ambiguous between the following readings.

~∃x(~[Px & Sx] & Dx)

there is no thing x such that
x is not a Poisonous Snake,
but x is Dangerous

$\sim\exists x([\sim Px \ \& \ Sx] \ \& \ Dx)$

there is no thing x such that
x is a nonPoisonous Snake,
but x is Dangerous

To see that the original sentence really is ambiguous, consider the following four (very short) paragraphs.

(1) Few snakes are dangerous. In fact,
only poisonous snakes are dangerous.

(2) Few reptiles are dangerous. In fact,
only poisonous snakes are dangerous.

(3) Few animals are dangerous. In fact,
only poisonous snakes are dangerous.

(4) Few things are dangerous. In fact,
only poisonous snakes are dangerous.

In each paragraph, when we get to the second sentence, it is clear what the topic is – snakes, reptiles, animals, or things in general. What the topic is helps to determine the meaning of the second sentence.

For example, in the first paragraph, by the time we get to the second sentence, it is clear that we are talking *exclusively* about snakes, and not things in general. In particular, the sentence does not say whether there are any dangerous tigers, or dangerous mushrooms.

By contrast, in the fourth paragraph, the first sentence makes it clear that we are talking about things in general, so the second sentence is intended to exclude from the class of dangerous things anything that is not a poisonous snake.

An alternative method of clarifying the topic of the sentence is to rewrite the four sentences as follows.

(1) only Poisonous Snakes are Dangerous snakes
(2) only Poisonous Snakes are Dangerous reptiles
(3) only Poisonous Snakes are Dangerous animals
(4) only Poisonous Snakes are Dangerous things

These may be straightforwardly paraphrased and symbolized as follows.

(0) only A are B
no non A is B
$\sim\exists x(\sim Ax \ \& \ Bx)$

(1) only PS are DS
no non(PS) is DS
$\sim\exists x(\sim[Px \,\&\, Sx] \,\&\, [Dx \,\&\, Sx])$

(2) only PS are DR
no non(PS) is DR
$\sim\exists x(\sim[Px \,\&\, Sx] \,\&\, [Dx \,\&\, Rx])$

(3) only PS are DA
no non(PS) is DA
$\sim\exists x(\sim[Px \,\&\, Sx] \,\&\, [Dx \,\&\, Ax])$

(4) only PS are D
no non(PS) is D
$\sim\exists x(\sim[Px \,\&\, Sx] \,\&\, Dx)$

If we prefer to use the ‘every’ paraphrase of ‘only’, then the paraphrase and symbolization goes as follows.

(0) only A are B
every non A is non B
$\forall x(\sim Ax \rightarrow \sim Bx)$

(1) only PS are DS
every non(PS) is non(DS)
$\forall x(\sim[Px \,\&\, Sx] \rightarrow \sim[Dx \,\&\, Sx])$

(2) only PS are DR
every non(PS) is non(DR)
$\forall x(\sim[Px \,\&\, Sx] \rightarrow \sim[Dx \,\&\, Rx])$

(3) only PS are DA
every non(PS) is non(DA)
$\forall x(\sim[Px \,\&\, Sx] \rightarrow \sim[Dx \,\&\, Ax])$

(4) only PS are D
every non(PS) is non-D
$\forall x(\sim[Px \,\&\, Sx] \rightarrow \sim Dx)$

17. ‘THE ONLY’

The subtleties of ‘only’ are further complicated by combining it with the word ‘the’ to produce ‘the only’.

[Still more complications arise when ‘the’ is combined with ‘only’ (‘all’) to produce ‘only the’ (‘all the’); however, we are only going to deal with ‘the only’.]

The nice thing about ‘the only’ is that it enables us to make ‘only’ statements without the kind of ambiguity seen in the previous section. Recall that only poisonous snakes are dangerous

is ambiguous between any of the following (among others):

only poisonous snakes are dangerous snakes
only poisonous snakes are dangerous reptiles
only poisonous snakes are dangerous animals
only poisonous snakes are dangerous things

These four propositions can also be expressed using ‘the only’, as follows.

(1) *the only* dangerous **snakes** are poisonous **snakes**
or: poisonous **snakes** are *the only* dangerous **snakes**

(2) *the only* dangerous **reptiles** are poisonous **snakes**
or: poisonous **snakes** are *the only* dangerous **reptiles**

(3) *the only* dangerous **animals** are poisonous **snakes**
or: poisonous **snakes** are *the only* dangerous **animals**

(4) *the only* dangerous **things** are poisonous **snakes**
or: poisonous **snakes** are *the only* dangerous **things**

The general form of these is:

the only AB are CD
or: CD are the only AB

Here, ‘AB’ and ‘BC’ are conjunctively-combined predicates.

Certain simplifications occasionally occur. For example, B and D may be the same predicate, or B may be the vacuous predicate ‘is a thing’ (which is never explicitly symbolized, since everything is a thing!).

The paraphrase and symbolization of ‘the only’ statements follows a pattern similar to the paraphrase and symbolization of ‘only’ statements. In particular, the paraphrase utilizes both ‘no’ and ‘not’. However, the details are importantly different.

Recall that

only A are B

is paraphrased:

no non A are B

Statements involving ‘the only’ are similarly paraphrased; specifically,

the only AB are CD
CD are the only AB

are paraphrased:

no AB are not CD

So, for example, we have the following paraphrases and symbolizations of (1)-(4).

(1) the only dangerous snakes are poisonous snakes
no dangerous snakes are not poisonous snakes
no DS are not PS
$\sim\exists x([Dx \& Sx] \& \sim[Px \& Sx])$
or: the only dangerous snakes are poisonous
no dangerous snakes are not poisonous
no DS are not P
$\sim\exists x([Dx \& Sx] \& \sim Px)$

(2) the only dangerous reptiles are poisonous snakes
no dangerous reptiles are not poisonous snakes
no DR are not PS
$\sim\exists x([Dx \& Rx] \& \sim[Px \& Sx])$

(3) the only dangerous animals are poisonous snakes
no dangerous animals are not poisonous snakes
no DS are not PS
$\sim\exists x([Dx \& Ax] \& \sim[Px \& Sx])$

(4) the only dangerous things are poisonous snakes
no dangerous things are not poisonous snakes
no D are not PS
$\sim\exists x(Dx \& \sim[Px \& Sx])$

Two features of the above should be noted, about (1) and (4). Both involve situations in which only three predicates are involved. In (1), the predicate 'is a snake' is repeated, and is equivalent to the sentence in which the second occurrence is simply dropped. In particular,

the only AB are CB

is equivalent to

the only AB are C,

which is paraphrased and symbolized:

no AB are not C
$\sim\exists x([Ax \& Bx] \& \sim Cx)$

In (4), the predicate 'is a thing' is vacuous; hence, it is not symbolized. In particular,

the only A things are CD

is equivalent to

the only A are CD,

which is paraphrased and symbolized:

no A are not CD
$\sim\exists x(Ax \;\&\; \sim[Cx \;\&\; Dx])$.

Note: Students who seek the *shortest symbolization* of a given statement may wish to consider the following equivalent symbolization. Recall that

no A are not B $\sim\exists x(Ax \;\&\; \sim Bx)$

is equivalent to

every A is B $\forall x(Ax \to Bx)$

Accordingly,

the only AB are CD,

which is paraphrased:

no AB are not CD $\sim\exists x([Ax \;\&\; Bx] \;\&\; \sim[Cx \;\&\; Dx])$

may also be paraphrased:

every AB is CD $\forall x([Ax \;\&\; Bx] \to [Cx \;\&\; Dx])$

Both symbolizations count as correct symbolizations; however, only the double-negative symbolizations will be given in the answers to the exercises.

18. DISJUNCTIVE COMBINATIONS OF PREDICATES

In Section 13, we examined many conjunctive predicate combinations, ones that may be symbolized by conjunctions. The curious thing about the logical structure of English is that often the word ‘and’, our archetypical word for conjunction, is used in a manner that does not allow it to be mechanically translated as a conjunction.

Consider the following two examples.

all Cats and Dogs are Suitable pets
only Cats and Dogs are Suitable pets

First, notice that ‘suitable’ does not combine conjunctively; for example, a suitable pet is (usually!) quite different from a suitable meal. We must accordingly treat the predicate combination ‘suitable pet’ as simple: ‘Sx’ stands for ‘x is a suitable pet’.

Let us concentrate on the first one for a moment. As a first attempt at translation, let us consider the following.

$\forall x([Cx \;\&\; Dx] \to Sx)$ WRONG!!!

What is wrong with this translation? Well, translating it back into English, piece by piece, yields the following:

for any thing x,
    if x is a cat
        and x is a dog,
            then x is a suitable pet

in other words,

for any thing x,
    if x is both a cat and a dog,
        then x is a suitable pet

This is surely true, but only because nothing is both a cat and a dog! By contrast, the original sentence is false, since cats and dogs do not *all* make suitable pets; many are not house-trained, many have rabies, etc.

The above translation is quite amusing, but nevertheless wrong. What is the correct translation? In particular, how does the word ‘and’ operate in the above sentence? One possible way to interpret ‘and’ as a genuine conjunction is to transform the original sentence into the following equivalent sentence.

all Cats are Suitable pets,
and
all Dogs are Suitable pets

This sentence is a conjunction, which is symbolized as follows.

$\forall x(Cx \rightarrow Sx)$ & $\forall x(Dx \rightarrow Sx)$

This formula involves two quantifiers; multiply-quantified formulas are the topic of a later section (Section 19). On the other hand, this formula is logically equivalent to the following singly-quantified formula.

$\forall x([Cx \vee Dx] \rightarrow Sx)$,

which reads:

for any thing x:
    if x is a cat
        or x is a dog,
            then x is a suitable pet

Thus, in some sense, to be explained shortly, the word ‘and’ is translated as a disjunction in this sentence.

In order to more fully understand what is going on, let us consider the second example.

only Cats and Dogs are Suitable pets

First, let us apply our earlier technique, transforming this sentence into the corresponding conjunction.

only Cats are Suitable pets, **and**
only Dogs are Suitable pets

As you can see, the simple transformation technique has failed, since the latter sentence is certainly not equivalent to the original. For, unlike the original sentence, the latter implies that any suitable pet is both a cat and a dog!

O.K., the first technique doesn't work. What about the second technique, which involves symbolizing the sentence using disjunction rather than conjunction? Let's see if this surprise attack will also work on the second example.

First, the overall form is:

only $\mathcal{A}$ are S,

where '$\mathcal{A}$' stands for 'Cats and Dogs'.

Its overall symbolization is therefore (using the ∀-version on 'only'):

$\forall x({\sim}\mathcal{A}x \rightarrow {\sim}Sx)$

Next, we propose the following disjunctive analysis of the pseudo-atomic formula 'Ax':

$\mathcal{A}x \;::\; [Cx \vee Dx]$

Thus, the final proposed symbolization is:

$\forall x({\sim}[Cx \vee Dx] \rightarrow {\sim}Sx)$.

Recalling that the negation of 'either-or' is 'neither-nor', this formula reads:

for any thing x:
if x is neither a Cat nor a Dog,
then x is not a Suitable pet

This is equivalent to:

for any thing x:
if x is a Suitable pet,
then x is either a Cat or a Dog

This seems to be a suitable translation of the original sentence.

The disjunction-approach seems to work. But how can one logically say that sometimes 'and' is translated as disjunction, when usually it is translated as conjunction? This does not make sense, unless we can tell when 'and' is conjunction, and when 'and' is disjunction.

As usual in natural language, the underlying logico-grammatical laws/rules are incredibly complex. But let us see if we can make a small amount of sense out of 'and'.

The key may lie in the distinction between singular and plural terms. Whereas predicate logic uses singular terms exclusively, natural English uses plural terms just as frequently as singular terms. The problem is in translating from plural-talk to singular-talk.

For example, the expressions,

cats, dogs, cats and dogs, suitable pets

are all plural terms; each one refers to a *class* or *set*. Let us name these classes as follows.

C: the class of all cats
D: the class of all dogs
E: the class of all cats and dogs
S: the class of all suitable pets

Now, let us consider the associated sentences. First, the sentences

all cats are suitable pets
all dogs are suitable pets
all cats and dogs are suitable pets

may be understood as asserting the following, respectively.

every member of class C (i.e., cats)
is also a member of class S (i.e., suitable pets)

every member of class D (i.e., dogs)
is also a member of class S (i.e., suitable pets)

every member of class E (i.e., cats and dogs)
is also a member of class S (i.e., suitable pets)

The notion of membership in a class is fairly straightforward in most cases. In particular, we have the following equivalences.

x is a member of C	::	x is a cat	::	Cx
x is a member of D	::	x is a dog	::	Dx
x is a member of S	::	x is a suitable pet	::	Sx

But the key equivalence concerns the class E, cats-and-dogs, which is given as follows.

x is a member of E	::	x is a cat **or** x is a dog	::	$[Cx \vee Dx]$

In other words, to say that x is a member of the class cats-and-dogs is to say that x is a cat *or* x is a dog (it surely is not to say that x is both a cat and a dog!). If x is a cat *or* x is a dog, then x is in the class cats-and-dogs; conversely, if x is in the class cats-and-dogs, then x is a cat *or* x is a dog.

So, when we translate the above sentences, using the above equivalences, we obtain:

for any x, if x is a cat,
then x is a suitable pet;
$\forall x(Cx \rightarrow Sx)$

for any x, if x is a dog,
then x is a suitable pet;
$\forall x(Dx \rightarrow Sx)$

for any x, if x is a cat *or* x is a dog,
then x is a suitable pet;
$\forall x([Cx \vee Dx] \rightarrow Sx)$

Now let's go back and do the example involving ‘only’.

only cats and dogs are suitable pets

which may be paraphrased as:

only members of E are members of S,

which is symbolized as:

$\forall x(\sim Ex \rightarrow \sim Sx)$

But ‘Ex’ means ‘x is a member of the class cats-and-dogs’, which means ‘x is a cat *or* x is a dog’, so we have as our final symbolization:

$\forall x(\sim[Cx \vee Dx] \rightarrow \sim Sx)$

Let us try one last example in this section.

the only mammals that are suitable pets are cats and dogs.

Once again, we have the compound-class expression ‘cats and dogs’. The overall form is

the only M that are S are E,

which we know can be symbolized in a number of ways, including the following.

$\sim\exists x([Mx \;\&\; Sx] \;\&\; \sim Ex)$
$\forall x([Mx \;\&\; Sx] \rightarrow Ex)$

But ‘Ex’ is short for [Cx ∨ Dx], so substituting back in, we obtain:

$\sim\exists x([Mx \;\&\; Sx] \;\&\; \sim[Cx \vee Dx])$
$\forall x([Mx \;\&\; Sx] \rightarrow [Cx \vee Dx])$

which are read as follows.

it is not true that:
 there is something x such that:
  it is a mammal and it is a suitable pet,
   but it is neither a cat nor a dog

for any thing x,
 if x is a mammal and x is a suitable pet,
  then x is a cat or x is a dog

19. MULTIPLE QUANTIFICATION IN MONADIC PREDICATE LOGIC

So far, we have concentrated on quantified formulas and negations of quantified formulas. A quantified formula is a formula whose principal connective is either a universal or an existential quantifier.

The grammar of predicate logic includes the grammar of sentential logic. In other words, when one has one or more predicate logic formulas, then one can combine them with sentential connectives in order to form more complex formulas. For example, if one has quantified formulas or negated quantified formulas $\mathcal{A}$ and $\mathcal{B}$, then one can combine them using conjunction (&), disjunction ($\vee$), conditional ($\rightarrow$), and biconditional ($\leftrightarrow$).

Consider the following formulas, together with possible English translations.

(1a)	$\forall$xFx	everyone is friendly
(2a)	$\exists$xFx	someone is friendly
(3a)	$\exists$x~Fx	someone is unfriendly
(4a)	~$\exists$xFx	no one is friendly
(5a)	$\forall$x~Fx	everyone is unfriendly
(6a)	~$\forall$xFx	not everyone is friendly

(1b)	$\forall$xHx	everyone is happy
(2b)	$\exists$xHx	someone is happy
(3b)	$\exists$x~Hx	someone is unhappy
(4b)	~$\exists$xHx	no one is happy
(5b)	$\forall$x~Hx	everyone is unhappy
(6b)	~$\forall$xHx	not everyone is happy

We can take any two of the above formulas (sentences) and combine them with any two-place connective. For example, we can combine them with conjunction. The following are a few examples.

(c1)	$\forall$xFx & $\forall$xHx	everyone is friendly, and everyone is happy
(c2)	$\forall$xFx & ~$\forall$xHx	everyone is friendly, but not everyone is happy
(c3)	$\exists$xHx & $\exists$x~Hx	someone is happy, and someone is unhappy
(c4)	~$\exists$xFx & $\forall$xHx	no one is friendly, but everyone is happy

Similarly, we can combine any pair of the above formulas (sentences) with the conditional connective. The following are a few examples.

(c5)	$\forall$xFx $\rightarrow$ $\forall$xHx	if everyone is friendly, then everyone is happy
(c6)	$\exists$xFx $\rightarrow$ $\exists$xHx	if someone is friendly, then someone is happy
(c7)	~$\exists$xFx $\rightarrow$ $\forall$x~Hx	if no one is friendly, then everyone is unhappy
(c8)	$\exists$x~Fx $\rightarrow$ ~$\exists$xHx	if someone is unfriendly, then no one is happy

At this point, probably the most important thing to recognize is the novelty of the above formulas. They are unlike any formula we have discussed so far. In particular, each one involves two quantifier expressions, whereas every previous example has involved at most one quantifier.

Let us pursue the difference for a moment. Consider the following pair of formulas.

(u1) $\forall x(Fx \rightarrow Hx)$
(u2) $\forall xFx \rightarrow \forall xHx$

They read as follows.

(r1) everything is such that:
  if it is F, then it is H

  every F is H

(r2) if everything is such that it is F,
  then everything is such that it is H

  if everything is F,
  then everything is H

What is the logical relation between (r1) and (r2)? Well, they are *not* equivalent; although (r1) implies (r2), (r2) does *not* imply (r1).

To see that (r2) does not imply (r1), consider the following counter-example to the argument form.

  if everyone is a Freshman, then everyone is happy
  therefore, every Freshman is happy

First, this concrete argument has the right form. Furthermore, the conclusion is false. So, what about the premise? This is a conditional; the antecedent is 'everyone is a Freshman'; this is false; the consequent is 'everyone is happy'; this is also false. Therefore, recalling the truth table for arrow (F→F=T), the conditional is true.

Whereas this argument is invalid, its converse is valid, but not sound. Its validity will be demonstrated in a later chapter.

Let us consider another example of the difference between a singly-quantified formula and a similar-looking multiply-quantified formula. Consider the following pair.

(e1) $\exists x(Fx \ \&\ Hx)$
(e2) $\exists xFx \ \&\ \exists xHx$

The colloquial readings are given as follows.

(c1) something is both F and H [or: some F is H]
(c2) something is F, and something is H

Once again the formulas are not logically equivalent; however, (c1) does imply (c2). For suppose that something is both F and H; then, it is F, and hence something is F; furthermore, it is H, and hence something is H. Hence, something is

F, and something is H. [We will examine this style of reasoning in detail in the chapter on derivations in predicate logic.]

So (c1) implies (c2). In order to see that (c2) does not imply (c1), consider the following counterexample.

someone is female, and someone is male
therefore, someone is both male and female

The premise is surely true, but the conclusion is false. Legally, if not biologically, everyone is *exclusively* male *or* female; no one is both male and female.

Having seen the basic theme (namely, combining quantified formulas with sentential connectives), let us now consider the three most basic variations on this theme.

First, one can combine the simple quantified formulas, listed above, using *non-standard* connectives ('unless', 'only if', etc.) Second, one can combine more complex quantified formulas (every A is B, every AB is C, etc.) using standard connectives. Finally, one can combine complex quantified formulas using non-standard connectives.

The following are examples of these three variations

(1a) everyone is happy, only if everyone is friendly
(1b) no one is happy, unless everyone is friendly
(2a) if every student is happy, then every Freshman is happy
(2b) every Freshman is a student, but not every student is a Freshman
(3a) every Freshman is Happy, only if every student is happy
(3b) no Student is happy, unless every student is friendly

Now, in translating English statements like the above, which involve more than one quantifier, and one or more *explicit* statement connectives, the best strategy is the following.

(1) Identify the overall sentential structure; i.e., identify the explicit sentential connectives;
(2) Identify the various (quantified) parts;
(3) Symbolize the overall sentential structure;
(4) Symbolize each (quantified) part;
(5) Substitute the symbolized parts into the overall sentential form.

This is pretty much the same strategy as for sentential symbolizations. The key difference is that, whereas in sentential logic one combines atomic formulas (capital letters), in predicate logic one combines quantified formulas as well.

With this strategy in mind, let us go back to the above examples.

Example 1

(1a) everyone is happy, only if everyone is friendly

The overall form of this sentence is:

$\mathcal{A}$ only if $\mathcal{B}$,

which is symbolized:

$\sim\mathcal{B} \rightarrow \sim\mathcal{A}$

The parts, and their respective symbolizations, are:

$\mathcal{A}$:	everyone is happy	$\forall$xHx
$\mathcal{B}$:	everyone is friendly	$\forall$xFx

So the final symbolization is:

$\sim\forall xFx \rightarrow \sim\forall xHx$

Example 2

(1b) no one is happy, unless everyone is friendly

The overall form is

$\mathcal{A}$ unless $\mathcal{B}$,

which is symbolized:

$\sim\mathcal{B} \rightarrow \mathcal{A}$

The parts, and their respective symbolizations, are:

$\mathcal{A}$:	no one is happy	$\sim\exists$xHx
$\mathcal{B}$:	everyone is friendly	$\forall$xFx

So the final symbolization is:

$\sim\forall xFx \rightarrow \sim\exists xHx$

Example 3

(2a) if every student is happy, then every Freshman is happy

The overall form of this sentence is:

if $\mathcal{A}$, then $\mathcal{B}$,

which is symbolized

$\mathcal{A} \rightarrow \mathcal{B}$

The parts, and their respective symbolizations, are:

$\mathcal{A}$:	every student is happy	$\forall x(Sx \rightarrow Hx)$
$\mathcal{B}$:	every Freshman is happy	$\forall x(Fx \rightarrow Hx)$

So the final symbolization is:

$\forall x(Sx \rightarrow Hx) \rightarrow \forall x(Fx \rightarrow Hx)$

Example 4

(2b) every Freshman is a student, but not every student is a Freshman.

The overall form of this sentence is:

$\mathcal{A}$ but $\mathcal{B}$ (i.e., $\mathcal{A}$ and $\mathcal{B}$),

which is symbolized

$\mathcal{A}$ & $\mathcal{B}$.

The parts, and their respective symbolizations, are:

$\mathcal{A}$:	every Freshman is a student	$\forall x(Fx \rightarrow Sx)$
$\mathcal{B}$:	not every student is a Freshman	$\sim\forall x(Sx \rightarrow Fx)$

So the final symbolization is:

$\forall x(Fx \rightarrow Sx)$ & $\sim\forall x(Sx \rightarrow Fx)$

Example 5

(3a) every Freshman is Happy, only if every student is happy

The overall form of this sentence is:

$\mathcal{A}$ only if $\mathcal{B}$,

which is symbolized

$\sim\mathcal{B} \rightarrow \sim\mathcal{A}$.

The parts, and their respective symbolizations, are:

$\mathcal{A}$:	every Freshman is happy	$\forall x(Fx \rightarrow Hx)$
$\mathcal{B}$:	every student is happy	$\forall x(Sx \rightarrow Hx)$

So the final symbolization is:

$\sim\forall x(Sx \rightarrow Hx) \rightarrow \sim\forall x(Fx \rightarrow Hx)$

Example 6

(3b) no Student is happy, unless every student is friendly

The overall form of this sentence is:

$\mathcal{A}$ unless $\mathcal{B}$,

which is symbolized

$\sim\mathcal{B} \rightarrow \mathcal{A}$

The parts, and their respective symbolizations, are:

$\mathcal{A}$:	no student is happy	$\sim\exists x(Sx \,\&\, Hx)$
$\mathcal{B}$:	every student is friendly	$\forall x(Sx \rightarrow Fx)$

So the final symbolization is:

$$\sim\forall x(Sx \rightarrow Fx) \rightarrow \sim\exists x(Sx \,\&\, Hx)$$

These are examples of the basic variations on the basic theme. There are also more complicated variations available. But in attacking a sentence that has a combination of several quantifiers and one or more sentential connectives (perhaps non-standard), the strategy is the same as before.

20. 'ANY' AND OTHER WIDE SCOPE QUANTIFIERS

Some quantifier expressions are occasionally used in ways that lead to confusion in symbolization in predicate logic. The troublesome expressions are:

any, anything, anyone, a, some.

Let us consider 'anyone' first. Clearly, this quantifier expression is sometimes equivalent to 'everyone', as seen in the following examples.

(1a)	anyone can fix your car	$\forall xFx$
(1b)	everyone can fix your car	$\forall xFx$
(2a)	if Jones can fix your car, then anyone can (fix your car)	$Fj \rightarrow \forall xFx$
(2b)	if Jones can fix your car, then everyone can (fix your car)	$Fj \rightarrow \forall xFx$

Here, 'j' stands for 'Jones', 'F_' stands for '_ can fix your car', and '$\forall x$' stands for 'every *person* x is such that...'

So far, our working hypothesis is that 'anyone' and 'everyone' are completely interchangeable. However, this hypothesis is quickly refuted when we interchange the roles of antecedent and consequent in (2a) and (2b), in which case we obtain the following statements.

(a) if anyone can fix your car, then Jones can (fix your car)

(e) if everyone can fix your car, then Jones can (fix your car)

Clearly, these are not equivalent! Whereas the former sentence could very well be an ad in the yellow pages, bragging about Jones' mechanical abilities, the latter would be a truly stupid ad, since it merely states a logical truth – that Jones can fix your car supposing *everyone* can.

Now, the symbolization of (e) is straightforward, it is a conditional with 'everyone can fix your car' [symbolized: $\forall$xFx] as antecedent and with 'Jones can fix your car' [symbolized: Fj] as consequent. It is accordingly symbolized as follows.

(e') $\forall$xFx → Fj

Notice that the main connective is arrow, and not a universal quantifier; in particular, when we read it literally, it goes as follows.

if everyone is F, *then* j is F

But what happens if we get confused and put in parentheses, so that '$\forall$x' is the main connective, and not '→'? In that case, we obtain the following formula,

$\forall$x(Fx → Fj),

which says something quite different from (e); but what? Well, the main connective is '$\forall$x', so the literal reading goes as follows.

everyone is such that: if he/she is F, then j is F.

Every universal formula is, in effect, a shorthand expression for a (possibly infinite) list of formulas, one formula for every individual in the universe. For example,

$\forall$xFx

is short for the following list:

Fa
Fb
Fc
etc.

And,

$\forall$x(Fx → Gx)

is short for the following list:

Fa → Ga
Fb → Gb
Fc → Gc
etc.

So, following this same pattern, the formula in question,

$\forall$x(Fx → Fj)

is short for the following list:

Fa → Fj
Fb → Fj
Fc → Fj
etc.

This list says, using the original scheme of abbreviation:

> if a can fix your car, then Jones can
> if b can fix your car, then Jones can
> if c can fix your car, then Jones can
> etc.

In other words,

> if *anyone* can fix your car, then Jones can

This sentence, of course, is one of our original sentences, which we now see is symbolized in predicate logic as follows.

> $\forall x(Fx \rightarrow Fj)$

In other words, although the English sentence *looks like* a conditional with 'anyone can fix your car' as its antecedent, in actuality, the sentence is a universal conditional. Although 'if...then...' appears to be the main connective, in fact 'anyone' is the main connective.

Consider another pair of examples involving 'any' versus 'every'.

(e) Jones does not know *everyone*

(a) Jones does not know *anyone*

As in the earlier case, 'everyone' and 'anyone' are not interchangeable. Whereas (e) is a negation of a universal, (a) is just the opposite, being a universal of a negation. The following are the respective symbolizations in *monadic* predicate logic, followed by their respective readings.

(e') $\sim\forall xKx$

> it is not true that
> everyone is such that
> Jones knows him/her.

(a') $\forall x\sim Kx$

> everyone is such that
> it is not true that
> Jones knows him/her.

Another way to express the latter is:

(a'') Jones knows no one.

Note: 'Kx' stands for 'Jones knows x', or 'x is known by Jones'. This can be further analyzed using a two place predicate '...knows...'; however, this further analysis is unnecessary to make the point about the difference between 'any' and 'every'.

The moral concerning ‘any’ versus ‘every’ seems to be this: On the one hand, the apparent grammatical position of ‘every’ coincides with its true logical position, in a sentence. On the other hand, the apparent grammatical position of ‘any’ does not coincide with its true logical position in a sentence. In particular, ‘any’ appears to be deeper inside the sentence than the affiliated sentential connectives, but its actual logical position is at the outside of the sentence. In short:

> The scope of ‘any’ is wide.
>
> The scope of ‘every’ is narrow.

Now what is worse is that ‘any’ is not the only wide-scope universal quantifier used in English; there are others, as witnessed by the following examples.

if a skunk enters, then every person will leave
if a skunk enters, then it won't be welcomed
a number is even if and only if it is divisible by 2
if someone were to enter, he/she would be surprised

We will deal with these particular examples shortly. First, let's consider what the problem might be. Clearly, both ‘a’ and ‘some’ are occasionally used as existential quantifiers; for example,

a tree grows in Brooklyn,

and

some tree grows in Brooklyn

both mean

at least one tree grows in Brooklyn,

which may be paraphrased as

there is at least one thing such that
it is a tree
and it grows in Brooklyn,

which is symbolized (in monadic logic, at least) as follows:

$\exists x(Tx \& Gx)$

But what if I say

if a tree grows in Brooklyn, then it is sturdy

This is a *much* harder symbolization problem! The problem is how do the quantifier ‘a’, the pronoun ‘it’, and the connective ‘if-then’ interact logically.

Consider an analogous example. which might be clearer.

if a number is divisible by 2, then it is even.

Here, we are clearly not talking about some *particular* number, which is even if it is divisible by 2; rather, we are talking about every/any number. In particular, this sentence can be paraphrased as

any number that is divisible by 2 is even,

or

every number is such that:
 if it is divisible by 2,
  then it is even.

These are symbolized as follows,

$\forall x(Dx \rightarrow Ex)$,

where '$\forall x$' means 'every *number* is such that' or 'for any *number*'.

Going back to the Brooklyn tree example, it is symbolized in a parallel manner,

$\forall x(Gx \rightarrow Sx)$,

where, in this case, '$\forall x$' means 'every tree is such that' or 'for any tree'.

every tree is such that:
 if it grows in Brooklyn,
  then it is sturdy

Now let us symbolize the earlier sentences.

if a skunk enters, then every person will leave

$\forall x(Sx \rightarrow [Ex \rightarrow \forall x(Px \rightarrow Lx)])$

if a skunk enters, then it won't be welcomed

$\forall x(Sx \rightarrow [Ex \rightarrow \sim Wx])$

a number is even if and only if it is divisible by 2

$\forall x(Nx \rightarrow [Ex \leftrightarrow Dx])$

if someone were to enter, he/she would be surprised

$\forall x(Ex \rightarrow Sx)$

By way of concluding this section, we observe that in certain *special* circumstances sentences containing wide-scope universal quantifiers ('a', 'any', etc.) can be translated into corresponding sentences containing narrow-scope existential quantifiers.

Let us go back to the example concerning the mechanic Jones.

if anyone can fix your car, then Jones can (fix your car).

One way to look at this is by way of a round-about paraphrase that goes as follows.

if Jones can*not* fix your car, then no one can (fix your car)

This is, just as it appears, a conditional, which is symbolized as follows.

$\sim Fj \rightarrow \sim\exists xFx$

if j is not F, then no one is F

Now, you will recall the following equivalence of sentential logic:

$\sim\mathcal{A} \rightarrow \sim\mathcal{B} \;::\; \mathcal{B} \rightarrow \mathcal{A}$

Accordingly, the above formula is equivalent to the following formula.

$\exists xFx \rightarrow Fj$

which translates into colloquial English as follows.

if *someone* can fix your car, then Jones can (fix your car).

This is consistent with our original symbolization of the sentence, since the following is an equivalence of predicate logic (as we will be able to demonstrate in a later chapter!)

$\forall x(Fx \rightarrow Fj) \;::\; \exists xFx \rightarrow Fj$

This is a special case of a more general scheme given as follows.

$\forall x(F[x] \rightarrow \mathcal{B}) \;::\; \exists xF[x] \rightarrow \mathcal{B}$

Here, F[x] is any formula in which ‘x’ occurs "free", and $\mathcal{B}$ is any formula in which ‘x’ is does not occur "free" (Consult later appendix concerning freedom and bondage of variables.)

Rather than dwell on the general problem, let us consider a few special cases. First, let us do an example contrasting ‘if every...’ and ‘if any...’.

if everyone fails the exam, then everyone will be sad

if anyone fails the exam, then everyone will be sad

Whereas ‘everyone’ is a narrow-scope universal quantifier, ‘anyone’ is a wide-scope universal quantifier, so the symbolizations go as follows.

$\forall xFx \rightarrow \forall xSx$

$\forall x(Fx \rightarrow \forall xSx)$

Remember, the latter is short for the following (possibly infinite) list.

$Fa \to \forall xSx$	if a fails, then everyone will be sad
$Fb \to \forall xSx$	if b fails, then everyone will be sad
$Fc \to \forall xSx$	if c fails, then everyone will be sad
$Fd \to \forall xSx$	if d fails, then everyone will be sad
etc.	

Now, in the formula

$\forall x(Fx \to \forall xSx)$,

'x' is free in 'Fx', but 'x' is not free in '$\forall xSx$', so we can apply the above-mentioned equivalence, to obtain:

$\exists xFx \to \forall xSx$,

which reads

if someone fails, then everyone will be sad

But what about the following:

if anyone fails the exam, he/she will be sad

This is symbolized the same as any 'if any...' statement:

$\forall x(Fx \to Sx)$,

which is short for the following (infinite) list:

$Fa \to Sa$	if a fails, then a will be sad
$Fb \to Sb$	if b fails, then b will be sad
$Fc \to Sc$	if c fails, then c will be sad
etc.	

This is *not* equivalent to a corresponding conditional with a narrow-scope existential quantifier, for example,

$\exists xFx \to Sx$,

which is equivalent to

$\exists xFx \to Sy$,

which reads:

if someone fails, then this (person) will be sad,

where 'this' points at whomever the person speaking chooses.

21. EXERCISES FOR CHAPTER 6

Directions for every exercise set:

Using the suggested abbreviations (the capitalized words), translate each of the following into the language of predicate logic.

EXERCISE SET A

1. JAY is a FRESHMAN.
2. KAY is a JUNIOR.
3. JAY and KAY are STUDENTS.
4. JAY is TALLER than KAY.
5. JAY is not SMARTER than KAY.
6. FRAN INTRODUCED JAY to KAY.
7. FRAN did not INTRODUCE KAY to JAY.
8. CHRIS is TALLER than both JAY and KAY.
9. JAY and KAY are MARRIED (to each other).
10. Both JAY and KAY are MARRIED.
11. Neither JAY nor KAY is MARRIED.
12. Although JAY and KAY are both MARRIED, they are not MARRIED to each other.
13. Neither JAY nor KAY is a SENIOR.
14. If JAY is a SOPHOMORE, then so is KAY.
15. If JAY and KAY LIVE off-campus, then neither of them is a FRESHMAN.
16. If neither JAY nor KAY is a FRESHMAN, then both of them are SOPHOMORES.
17. JAY and KAY are not ROOMMATES unless they are MARRIED.
18. JAY or KAY is the STUDENT body president, but not both.
19. JAY and KAY are FRIENDS if and only if they are ROOMMATES.
20. JAY and KAY are neither SIBLINGS nor COUSINS.

EXERCISE SET B

21. Everything is POSSIBLE.
22. Something is POSSIBLE.
23. Nothing is POSSIBLE.
24. Something is not POSSIBLE.
25. Not everything is POSSIBLE.
26. Everything is imPOSSIBLE.
27. Nothing is imPOSSIBLE.
28. Something is imPOSSIBLE.
29. Not everything is imPOSSIBLE.

30 Not a thing can be CHANGED.

31. Everyone is PERFECT.
32. Someone is PERFECT.
33. No one is PERFECT.
34. Someone is not PERFECT.
35. Not everyone is PERFECT.
36. Everyone is imPERFECT.
37. No one is imPERFECT.
38. Someone is imPERFECT.
39. Not everyone is imPERFECT
40. Not a single person CAME.

EXERCISE SET C

41. Every STUDENT is HAPPY.
42. Some STUDENT is HAPPY.
43. No STUDENT is HAPPY.
44. Some STUDENT is not HAPPY.
45. Not every STUDENT is HAPPY.
46. Every STUDENT is unHAPPY.
47. Some STUDENT is unHAPPY.
48. No STUDENT is unHAPPY.
49. Not every STUDENT is unHAPPY.
50. Not a single STUDENT is HAPPY.
51. All SNAKES HIBERNATE.
52. Some SENATORS are HONEST.
53. No SCOUNDRELS are HONEST.
54. Some SENATORS are not HONEST.
55. Not all SNAKES are HARMFUL.
56. All SKUNKS are unHAPPY.
57. Some SENATORS are unHAPPY.
58. No SCOUNDRELS are unHAPPY.
59. Not all SNAKES are unHAPPY.
60. Not a single SCOUNDREL is HONEST.

EXERCISE SET D

61. No one who is HONEST is a POLITICIAN.
62. No one who isn't COORDINATED is an ATHLETE.
63. Anyone who is ATHLETIC is WELL-ADJUSTED.
64. Everyone who is SENSITIVE is HEALTHY.
65. At least one ATHLETE is not BOORISH.
66. There is at least one POLITICIAN who is HONEST.
67. Everyone who isn't VACATIONING is WORKING.
68. Everything is either MATERIAL or SPIRITUAL.
69. Nothing is both MATERIAL and SPIRITUAL.
70. At least one thing is neither MATERIAL nor SPIRITUAL.

EXERCISE SET E

71. Every CLEVER STUDENT is AMBITIOUS.
72. Every AMBITIOUS STUDENT is CLEVER.
73. Every STUDENT is both CLEVER and AMBITIOUS.
74. Every STUDENT is either CLEVER or not AMBITIOUS.
75. Every STUDENT who is AMBITIOUS is CLEVER.
76. Every STUDENT who is CLEVER is AMBITIOUS.
77. Some CLEVER STUDENTS are AMBITIOUS.
78. Some CLEVER STUDENTS are not AMBITIOUS.
79. Not every CLEVER STUDENT is AMBITIOUS.
80. Not every AMBITIOUS STUDENT is CLEVER.
81. Some AMBITIOUS STUDENTS are not CLEVER.
82. No AMBITIOUS STUDENT is CLEVER.
83. No CLEVER STUDENT is AMBITIOUS.
84. No STUDENT is either CLEVER or AMBITIOUS.
85. No STUDENT is both CLEVER and AMBITIOUS.
86. Every AMBITIOUS PERSON is a CLEVER STUDENT.
87. No AMBITIOUS PERSON is a CLEVER STUDENT.
88. Some AMBITIOUS PERSONS are not CLEVER STUDENTS.
89. Not every AMBITIOUS PERSON is a CLEVER STUDENT.
90. Not all CLEVER PERSONS are STUDENTS.

EXERCISE SET F

91. Only MEMBERS are ALLOWED to enter.
92. Only CITIZENS who are REGISTERED are ALLOWED to vote.
93. The only non-MEMBERS who are ALLOWED inside are GUESTS.
94. DOGS are the only PETS worth having.
95. DOGS are not the only PETS worth having.
96. The only DANGEROUS SNAKES are the ones that are POISONOUS.
97. The only DANGEROUS things are POISONOUS SNAKES.
98. Only POISONOUS SNAKES are DANGEROUS (snakes).
99. Only POISONOUS SNAKES are DANGEROUS ANIMALS.
100. The only FRESHMEN who PASS intro logic are the ones who WORK.

EXERCISE SET G

101. All HORSES and COWS are FARM animals.
102. All CATS and DOGS make EXCELLENT pets.
103. RAINY days and MONDAYS always get me DOWN.
104. CATS and DOGS are the only SUITABLE pets.
105. The only PERSONS INSIDE are MEMBERS and GUESTS.
106. The only CATS and DOGS that are SUITABLE pets are the ones that have been HOUSE-trained.
107. CATS and DOGS are the only ANIMALS that are SUITABLE pets.
108. No CATS or DOGS are SOLD here.
109. No CATS or DOGS are SOLD, that are not VACCINATED.
110. CATS and DOGS that have RABIES are not SUITABLE pets.

EXERCISE SET H

111. If nothing is sPIRITUAL, then nothing is SACRED.
112. If everything is MATERIAL, then nothing is SACRED.
113. Not everything is MATERIAL, provided that something is SACRED.
114. If everything is SACRED, then all COWS are SACRED.
115. If nothing is SACRED, then no COW is SACRED.
116. If all COWS are SACRED, then everything is SACRED.
117. All FRESHMEN are STUDENTS, but not all STUDENTS are FRESHMEN.
118. If every STUDENT is CLEVER, then every FRESHMAN is CLEVER.
119. If every BIRD can FLY, then every BIRD is DANGEROUS.
120. If some SNAKE is not POISONOUS, then not every SNAKE is DANGEROUS.
121. No PROFESSOR is HAPPY, unless some STUDENTS are CLEVER.
122. All COWS are SACRED, only if no COW is BUTCHERED.
123. Some SNAKES are not DANGEROUS, only if some SNAKES are not POISONOUS.
124. If everything is a COW, and every COW is SACRED, then everything is SACRED.
125. If everything is a COW, and no COW is SACRED, then nothing is SACRED.
126. If every BOSTONIAN CAB driver is a MANIAC, then no BOSTONIAN PEDESTRIAN is SAFE.

127. If everyone is FRIENDLY, then everyone is HAPPY.

128. Unless every PROFESSOR is FRIENDLY, no STUDENT is HAPPY.

129. Every STUDENT is HAPPY, only if every PROFESSOR is FRIENDLY.

130. No STUDENT is unHAPPY, unless every PROFESSOR is unFRIENDLY.

EXERCISE SET I

131. If anyone is FRIENDLY, then everyone is HAPPY.

132. If anyone can FIX your car, then SMITH can.

133. If SMITH can't FIX your car, then no one can.

134. If everyone PASSES the exam, then everyone will be HAPPY.

135. If anyone PASSES the exam, then everyone will be HAPPY.

136. If everyone FAILS the exam, then no one will be HAPPY.

137. If anyone FAILS the exam, then no one will be HAPPY.

138. A SKUNK is DANGEROUS if and only if it is RABID.

139. If a CLOWN ENTERS the room, then every PERSON will be SURPRISED.

140. If a CLOWN ENTERS the room, then it will be DISPLEASED if no PERSON is SURPRISED.

22. ANSWERS TO EXERCISES FOR CHAPTER 6

Note: Only one translation is written down in each case; in most cases, there are alternative translations that are equally correct. Your translation is correct if and only if it is equivalent to the answer given below.

EXERCISE SET A

1. Fj
2. Jk
3. Sj & Sk
4. Tjk
5. ~Sjk
6. Ifjk
7. ~Ifkj
8. Tcj & Tck
9. Mjk
10. Mj & Mk
11. ~Mj & ~Mk
12. (Mj & Mk) & ~Mjk
13. ~Sj & ~Sk
14. Sj → Sk
15. (Lj & Lk) → (~Fj & ~Fk)
16. (~Fj & ~Fk) → (Sj & Sk)
17. ~Mjk → ~Rjk
18. (Sj ∨ Sk) & ~(Sj & Sk)
19. Fjk ↔ Rjk
20. ~Sjk & ~Cjk

EXERCISE SET B

21. ∀xPx
22. ∃xPx
23. ~∃xPx
24. ∃x~Px
25. ~∀xPx
26. ∀x~Px
27. ~∃x~Px
28. ∃x~Px
29. ~∀x~Px
30. ~∃xCx
31. ∀xPx
32. ∃xPx
33. ~∃xPx
34. ∃x~Px
35. ~∀xPx
36. ∀x~Px
37. ~∃x~Px
38. ∃x~Px
39. ~∀x~Px

40. $\sim\exists x Cx$

EXERCISE SET C

41. $\forall x(Sx \rightarrow Hx)$
42. $\exists x(Sx \,\&\, Hx)$
43. $\sim\exists x(Sx \,\&\, Hx)$
44. $\exists x(Sx \,\&\, \sim Hx)$
45. $\sim\forall x(Sx \rightarrow Hx)$
46. $\forall x(Sx \rightarrow \sim Hx)$
47. $\exists x(Sx \,\&\, \sim Hx)$
48. $\sim\exists x(Sx \,\&\, \sim Hx)$
49. $\sim\forall x(Sx \rightarrow \sim Hx)$
50. $\sim\exists x(Sx \,\&\, Hx)$
51. $\forall x(Sx \rightarrow Hx)$
52. $\exists x(Sx \,\&\, Hx)$
53. $\sim\exists x(Sx \,\&\, Hx)$
54. $\exists x(Sx \,\&\, \sim Hx)$
55. $\sim\forall x(Sx \rightarrow Hx)$
56. $\forall x(Sx \rightarrow \sim Hx)$
57. $\exists x(Sx \,\&\, \sim Hx)$
58. $\sim\exists x(Sx \,\&\, \sim Hx)$
59. $\sim\forall x(Sx \rightarrow \sim Hx)$
60. $\sim\exists x(Sx \,\&\, Hx)$

EXERCISE SET D

61. $\sim\exists x(Hx \,\&\, Px)$
62. $\sim\exists x(\sim Cx \,\&\, Ax)$
63. $\forall x(Ax \rightarrow Wx)$
64. $\forall x(Sx \rightarrow Hx)$
65. $\exists x(Ax \,\&\, \sim Bx)$
66. $\exists x(Px \,\&\, Hx)$
67. $\forall x(\sim Vx \rightarrow Wx)$
68. $\forall x(Mx \vee Sx)$
69. $\sim\exists x(Mx \,\&\, Sx)$
70. $\exists x(\sim Mx \,\&\, \sim Sx)$

EXERCISE SET E

71. $\forall x([Cx \,\&\, Sx] \rightarrow Ax)$
72. $\forall x([Ax \,\&\, Sx] \rightarrow Cx)$
73. $\forall x(Sx \rightarrow [Cx \,\&\, Ax])$
74. $\forall x(Sx \rightarrow [Cx \vee {\sim}Ax])$
75. $\forall x([Sx \,\&\, Ax] \rightarrow Cx)$
76. $\forall x([Sx \,\&\, Cx] \rightarrow Ax)$
77. $\exists x([Cx \,\&\, Sx] \,\&\, Ax)$
78. $\exists x([Cx \,\&\, Sx] \,\&\, {\sim}Ax)$
79. ${\sim}\forall x([Cx \,\&\, Sx] \rightarrow Ax)$
80. ${\sim}\forall x([Ax \,\&\, Sx] \rightarrow Cx)$
81. $\exists x([Ax \,\&\, Sx] \,\&\, {\sim}Cx)$
82. ${\sim}\exists x([Ax \,\&\, Sx] \,\&\, Cx)$
83. ${\sim}\exists x([Cx \,\&\, Sx] \,\&\, Ax)$
84. ${\sim}\exists x(Sx \,\&\, [Cx \vee Ax])$
85. ${\sim}\exists x(Sx \,\&\, [Cx \,\&\, Ax])$
86. $\forall x([Ax \,\&\, Px] \rightarrow [Cx \,\&\, Sx])$
87. ${\sim}\exists x([Ax \,\&\, Px] \,\&\, [Cx \,\&\, Sx])$
88. $\exists x([Ax \,\&\, Px] \,\&\, {\sim}[Cx \,\&\, Sx])$
89. ${\sim}\forall x([Ax \,\&\, Px]) \rightarrow [Cx \,\&\, Sx])$
90. ${\sim}\forall x([Cx \,\&\, Px] \rightarrow Sx)$

EXERCISE SET F

91. ${\sim}\exists x({\sim}Mx \,\&\, Ax)$
92. ${\sim}\exists x({\sim}[Cx \,\&\, Rx] \,\&\, Ax)$
93. ${\sim}\exists x([{\sim}Mx \,\&\, Ax] \,\&\, {\sim}Gx)$
94. ${\sim}\exists x(Px \,\&\, {\sim}Dx)$
95. $\exists x(Px \,\&\, {\sim}Dx)$
96. ${\sim}\exists x([Dx \,\&\, Sx] \,\&\, {\sim}Px)$
97. ${\sim}\exists x(Dx \,\&\, {\sim}[Px \,\&\, Sx])$
98. ${\sim}\exists x({\sim}[Px \,\&\, Sx] \,\&\, [Dx \,\&\, Sx])$
99. ${\sim}\exists x({\sim}[Px \,\&\, Sx] \,\&\, [Dx \,\&\, Ax])$
100. ${\sim}\exists x([Fx \,\&\, Px] \,\&\, {\sim}Wx)$

EXERCISE SET G

101. $\forall x([Hx \vee Cx] \rightarrow Fx)$
102. $\forall x([Cx \vee Dx] \rightarrow Ex)$
103. $\forall x([Rx \vee Mx] \rightarrow Dx)$
104. ${\sim}\exists x(Sx \,\&\, {\sim}[Cx \vee Dx])$
105. ${\sim}\exists x([Px \,\&\, Ix] \,\&\, {\sim}[Mx \vee Gx])$
106. ${\sim}\exists x(\{[Cx \vee Dx] \,\&\, Sx\} \,\&\, {\sim}Hx)$
107. ${\sim}\exists x([Ax \,\&\, Sx] \,\&\, {\sim}[Cx \vee Dx])$
108. ${\sim}\exists x([Cx \vee Dx] \,\&\, Sx)$
109. ${\sim}\exists x([(Cx \vee Dx) \,\&\, {\sim}Vx] \,\&\, Sx)$
110. $\forall x([(Cx \vee Dx) \,\&\, Rx] \rightarrow {\sim}Sx)$

EXERCISE SET H

111. ~∃xPx → ~∃xSx
112. ∀xMx → ~∃xSx
113. ∃xSx → ~∀xMx
114. ∀xSx → ∀x(Cx → Sx)
115. ~∃xSx → ~∃x(Cx & Sx)
116. ∀x(Cx → Sx) → ∀xSx
117. ∀x(Fx → Sx) & ~∀x(Sx → Fx)
118. ∀x(Sx → Cx) → ∀x(Fx → Cx)
119. ∀x(Bx → Fx) → ∀x(Bx → Dx)
120. ∃x(Sx & ~Px) → ~∀x(Sx → Dx)
121. ~∃x(Sx & Cx) → ~∃x(Px & Hx)
122. ∃x(Cx & Bx) → ~∀x(Cx → Sx)
123. ~∃x(Sx & ~Px) → ~∃x(Sx & ~Dx)
124. [∀xCx & ∀x(Cx → Sx)] → ∀xSx
125. [∀xCx & ~∃x(Cx & Sx)] → ~∃xSx
126. ∀x([Bx&Cx] → Mx) → ~∃x([Bx&Px] & Sx)
127. ∀xFx → ∀xHx
128. ~∀x(Px → Fx) → ~∃x(Sx & Hx)
129. ~∀x(Px → Fx) → ~∀x(Sx → Hx)
130. ~∀x(Px → ~Fx) → ~∃x(Sx & ~Hx)

EXERCISE SET I

131. ∀x(Fx → ∀xHx)
132. ∀x(Fx → Fs)
133. ~Fs → ~∃xFx
134. ∀xPx → ∀xHx
135. ∀x(Px → ∀xHx)
136. ∀xFx → ~∃xHx
137. ∀x(Fx → ~∃xHx)
138. ∀x(Sx → [Dx ↔ Rx])
139. ∀x{[Cx & Ex] → ∀x(Px → Sx)}
140. ∀x{[Cx & Ex] → [~∃y(Py & Sy) → Dx]}