9. CONDITIONAL DERIVATION

So far, we only have one method by which to cancel a show-line – direct derivation. In the present section, we examine a new derivation method, which will enable us to prove valid a larger class of sentential arguments.

Consider the following argument.

(A) \( P \rightarrow Q \)
(\( Q \rightarrow R \))

This argument is valid, as can easily be demonstrated using truth-tables. Can we derive the conclusion from the premises? The following begins the derivation.

(1) \( P \rightarrow Q \quad \text{Pr} \)
(2) \( Q \rightarrow R \quad \text{Pr} \)
(3) \( \neg \); \( P \rightarrow R \quad ??? \)
(4) \( ??? \quad ??? \)

What formulas can we write down at line (4)? There are numerous formulas that follow from the premises according to the inference rules. But, not a single one of them makes any progress toward showing the conclusion \( P \rightarrow R \). In fact, upon close examination, we see that we have no means at our disposal to prove this argument. We are stuck.

In other words, as it currently stands, derivation system SL is inadequate. The above argument is valid, by truth-tables, but it cannot be proven in system SL. Accordingly, system SL must be strengthened so as to allow us to prove the above argument. Of course, we don't want to make the system so strong that we can derive invalid conclusions, so we have to be careful, as usual.

How might we argue for such a conclusion? Consider a concrete instance of the argument form.

(I) if the gas tank gets a hole, then the car runs out of gas;
if the car runs out of gas, then the car stops;
therefore, if the gas tank gets a hole, then the car stops.

In order to argue for the conclusion of (I), it seems natural to argue as follows. First, suppose the premises are true, in order to show the conclusion. The conclusion says that

the car stops if the gas tank gets a hole

or in other words,

the car stops supposing the gas tank gets a hole.

So, suppose also that the antecedent,
the gas tank gets a hole,
is true.  In conjunction with the first premise, we can infer the following by \textit{modus ponens} (\(\rightarrow O\)):

the car runs out of gas.

And from this in conjunction with the second premise, we can infer the following by \textit{modus ponens} (\(\rightarrow O\)).

the car stops

So supposing the antecedent (the gas tank gets a hole), we have deduced the consequent (the car stops). In other words, we have shown the conclusion – if the gas tank gets a hole, then the car stops.

The above line of reasoning is made formal in the following official derivation.

\textbf{Example 1}

\begin{align*}
(1) & \quad H \rightarrow R & \text{Pr} \\
(2) & \quad R \rightarrow S & \text{Pr} \\
(3) & \quad \text{SHOW: } H \rightarrow S & \text{CD} \\
(4) & \quad H & \text{As} \\
(5) & \quad \text{SHOW: } S & \text{DD} \\
(6) & \quad R & 1,4,\rightarrow O \\
(7) & \quad S & 2,6,\rightarrow O
\end{align*}

This new-fangled derivation requires explaining. First of all, there are two show-lines; in particular, one derivation is nested inside another derivation. This is because the original problem – showing \(H \rightarrow S\) – is reduced to another problem, showing \(S\) assuming \(H\). This procedure is in accordance with a new show-rule, called conditional derivation, which may be intuitively formulated as follows.

\begin{center}
\textbf{Conditional Derivation} (Intuitive Formulation)
\end{center}

\begin{center}
In order to show a conditional \(A \rightarrow C\), it is sufficient to show the consequent \(C\), assuming the antecedent \(A\).
\end{center}

The official formulation of conditional derivation is considerably more complicated, being given by the following two system rules.
System Rule 5 (a show-rule)

**Conditional Derivation (CD)**

If one has a show-line of the form ‘SHOW: \( A \rightarrow C \)’, and one has \( C \) as a later available line, and there are no subsequent uncancelled show-lines, then one is entitled to box and cancel ‘SHOW: \( A \rightarrow C \)’.

The annotation is ‘CD’

System Rule 6 (an assumption rule)

If one has a show-line of the form ‘SHOW: \( A \rightarrow C \)’, then one is entitled to write down the antecedent \( A \) on the very next line, as an assumption.

The annotation is ‘As’

It is probably easier to understand conditional derivation by way of the associated picture.

This is supposed to depict the nature of conditional derivation; one shows a conditional \( A \rightarrow C \) by assuming its antecedent \( A \) and showing its consequent \( C \).

In order to further our understanding of conditional derivation, we do a few examples.
Example 2

(1) $P \rightarrow R$  Pr
(2) $Q \rightarrow S$  Pr
(3) \textbf{SHOW}: $(P \& Q) \rightarrow (R \& S)$  CD
(4) $P \& Q$  As
(5) \textbf{SHOW}: $R \& S$  DD
(6) $P$  4,\&O
(7) $Q$  4,\&O
(8) $R$  1,6,\rightarrow O
(9) $S$  2,7,\rightarrow O
(10) $R \& S$  8,9,\&I

Example 3

(1) $Q \rightarrow R$  Pr
(2) $R \rightarrow (P \rightarrow S)$  Pr
(3) \textbf{SHOW}: $(P \& Q) \rightarrow S$  CD
(4) $P \& Q$  As
(5) \textbf{SHOW}: $S$  DD
(6) $P$  4,\&O
(7) $Q$  4,\&O
(8) $R$  1,7,\rightarrow O
(9) $P \rightarrow S$  2,8,\rightarrow O
(10) $S$  6,9,\rightarrow O

The above examples involve two show-lines; each one involves a direct derivation inside a conditional derivation. The following examples introduce a new twist – three show-lines in the same derivation, with a conditional derivation inside a conditional derivation.

Example 4

(1) $(P \& Q) \rightarrow R$  Pr
(2) \textbf{SHOW}: $P \rightarrow (Q \rightarrow R)$  CD
(3) $P$  As
(4) \textbf{SHOW}: $Q \rightarrow R$  CD
(5) $Q$  As
(6) \textbf{SHOW}: $R$  DD
(7) $P \& Q$  3,5,\&I
(8) $R$  1,7,\rightarrow O
Example 5

(1) (P \& Q) \rightarrow R \quad \text{Pr}
(2) \text{SHOW: } (P \rightarrow Q) \rightarrow (P \rightarrow R) \quad \text{CD}
(3) P \rightarrow Q \quad \text{As}
(4) \text{SHOW: } P \rightarrow R \quad \text{CD}
(5) P \quad \text{As}
(6) \text{SHOW: } R \quad \text{DD}
(7) Q \quad 3,5,\rightarrow O
(8) P \& Q \quad 5,7,\& I
(9) R \quad 1,8,\rightarrow O

Needless to say, the depth of nesting is not restricted; consider the following example.

Example 6

(1) (P \& Q) \rightarrow (R \rightarrow S) \quad \text{Pr}
(2) \text{SHOW: } R \rightarrow [(P \rightarrow Q) \rightarrow (P \rightarrow S)] \quad \text{CD}
(3) R \quad \text{As}
(4) \text{SHOW: } (P \rightarrow Q) \rightarrow (P \rightarrow S) \quad \text{CD}
(5) P \rightarrow Q \quad \text{As}
(6) \text{SHOW: } P \rightarrow S \quad \text{CD}
(7) P \quad \text{As}
(8) \text{SHOW: } S \quad \text{DD}
(9) Q \quad 5,7,\rightarrow O
(10) P \& Q \quad 7,9,\& I
(11) R \rightarrow S \quad 1,10,\rightarrow O
(12) S \quad 3,11,\rightarrow O

Irrespective of the complexity of the above problems, they are solved in the same systematic manner. At each point where we come across ‘SHOW: A\rightarrow C’, we immediately write down two more lines – we assume the antecedent, A, in order to (attempt to) show the consequent, C.

That is all there is to it!
10. INDIRECT DERIVATION (FIRST FORM)

System SL is now a complete set of rules for sentential logic; every valid argument of sentential logic can be proved valid in system SL. System SL is also consistent, which is to say that no invalid argument can be proven in system SL. Demonstrating these two very important logical facts – that system SL is both complete and consistent – is well outside the scope of introductory logic. It rather falls under the scope of metalogic, which is studied in more advanced courses in logic.

Even though system SL is complete as it stands, we will nonetheless enhance it further, thereby sacrificing elegance in favor of convenience. Consider the following argument form.

\[(a1) \quad P \rightarrow Q \\
    \quad P \rightarrow \neg Q \]
\[
    \quad \neg P
\]

Using truth-tables, one can quickly demonstrate that (a1) is valid. What happens when we try to construct a derivation that proves it to be valid? Consider the following start.

\[(1) \quad P \rightarrow Q \quad \text{Pr} \\
(2) \quad P \rightarrow \neg Q \quad \text{Pr} \\
(3) \quad \neg : \neg P \quad ??? \\
(4) \quad ??? \quad ???
\]

An attempted derivation, using DD and CD, might go as follows.

Consider line (3), which is a negation. We cannot show it by conditional derivation; it's not a conditional! That leaves direct derivation. Well, the premises are both conditionals, so the appropriate rule is arrow-out. But arrow-out requires a minor premise. In the case of (1) we need P or \(~Q\); in the case of (2), we need P or \(\neg \neg Q\); none of these is available. We are stuck!

We are trying to show \(\neg P\), which says in effect that P is false. Let's try a sneaky approach to the problem. Just for the helluvit, let us assume the opposite of what we are trying to show, and see what happens. So right below ‘SHOW: \(\neg P\)’, we write P as an assumption. That yields the following partial derivation.

\[(1) \quad P \rightarrow Q \quad \text{Pr} \\
(2) \quad P \rightarrow \neg Q \quad \text{Pr} \\
(3) \quad \text{SHOW}: \neg P \quad ??? \\
(4) \quad P \quad \text{As??} \\
(6) \quad Q \quad 1,4,\rightarrow O \\
(7) \quad \neg Q \quad 1,5,\rightarrow O \\
(8) \quad Q \& \neg Q \quad 5,6,\& I
\]

We have gotten down to line (8) which is \(Q\&\neg Q\). From our study of truth-tables, we know that this formula is a self-contradiction; it is false no matter what. So we
see that assuming P at line (4) leads to a very bizarre result, a self-contradiction at line (8).

So, we have shown, in effect, that if P is true, then so is Q&~Q, which means that we have shown P→(Q&~Q). To see this, let us rewrite the problem as follows. Notice especially the new show-line (4).

\begin{align*}
(1) & \quad P \rightarrow Q \quad \text{Pr} \\
(2) & \quad P \rightarrow \neg Q \quad \text{Pr} \\
(3) & \quad \text{SHOW: } \neg P \quad ??? \\
(4) & \quad \text{SHOW: } P \rightarrow (Q \& \neg Q) \quad \text{CD} \\
(5) & \quad P \quad \text{As} \\
(6) & \quad \text{SHOW: } Q \& \neg Q \quad \text{DD} \\
(7) & \quad Q \quad 1,5,\rightarrow O \\
(8) & \quad \neg Q \quad 2,5,\rightarrow O \\
(9) & \quad Q \& \neg Q \quad 7,8,\&I \\
\end{align*}

This is OK as far as it goes, but it is still not complete; show-line (3) has not been cancelled yet, which is marked in the annotation column by ‘???’ Line (4) is permitted, by the show-line rule (we can try to show anything!). Lines (5) and (6) then are written down in accordance with conditional derivation. The remaining lines are completely ordinary.

So how do we complete the derivation? We are trying to show ~P; we have in fact shown P→(Q&~Q); in other words, we have shown that if P is true, then so is Q&~Q. But the latter can't be true, so neither can the former (by modus tollens). This reasoning can be made formal in the following part derivation.

\begin{align*}
(1) & \quad P \rightarrow Q \quad \text{Pr} \\
(2) & \quad P \rightarrow \neg Q \quad \text{Pr} \\
(3) & \quad \text{SHOW: } \neg P \quad \text{DD} \\
(4) & \quad \text{SHOW: } P \rightarrow (Q \& \neg Q) \quad \text{CD} \\
(5) & \quad P \quad \text{As} \\
(6) & \quad \text{SHOW: } Q \& \neg Q \quad \text{DD} \\
(7) & \quad Q \quad 1,5,\rightarrow O \\
(8) & \quad \neg Q \quad 2,5,\rightarrow O \\
(9) & \quad Q \& \neg Q \quad 7,8,\&I \\
(10) & \quad \neg (Q \& \neg Q) \quad ??? \\
(11) & \quad \neg P \quad 4,10,\rightarrow O \\
\end{align*}

This is an OK derivation, except for line (10), which has no justification. At this stage in the elaboration of system SL, we could introduce a new system rule that allows one to write ~(A&~A) at any point in a derivation. This rule would work perfectly well, but it is not nearly as tidy as what we do instead. We choose instead to abbreviate the above chain of reasoning considerably, by introducing a further show-rule, called indirect derivation, whose intuitive formulation is given as follows.
In order to show a negation $\sim \mathcal{A}$, it is sufficient to show any contradiction, assuming the un-negated formula, $\mathcal{A}$.

We must still provide the official formulation of indirect derivation, which as usual is considerably more complex; see below.

Recall that a contradiction is any formula whose truth table yields all F's in the output column. There are infinitely many contradictions in sentential logic. For this reason, at this point, it is convenient to introduce a new symbol into the vocabulary of sentential logic. In addition to the usual symbols – the letters, the connective symbols, and the parentheses – we introduce the symbol ‘$\Xi$’, in accordance with the following syntactic and semantic rules.

**Syntactic Rule:** $\Xi$ is a formula.

**Semantic Rule:** $\Xi$ is false no matter what.

[Alternatively, $\Xi$ is a "zero-place" logical connective, whose truth table always produces F.] In other words, $\Xi$ is a *generic contradiction*; it is equivalent to every contradiction.

With our new generic contradiction, we can reformulate Indirect Derivation as follows.

**Indirect Derivation** (First Form)

**Second Formulation**

In order to show a negation $\sim \mathcal{A}$, it is sufficient to show $\Xi$, assuming the un-negated formula, $\mathcal{A}$.

In addition to the syntactic and semantic rules governing $\Xi$, we also need inference rules; in particular, as with the other logical symbols, we need an elimination rule, and an introduction rule. These are given as follows.
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Contradiction -In (\(\text{xI}\))

\[
\begin{array}{c}
\text{A} \\
\sim \text{A} \\
\hline
\times
\end{array}
\]

Contradiction -Out (\(\text{xO}\))

\[
\begin{array}{c}
\times \\
\hline
\text{A}
\end{array}
\]

We will have little use for the elimination rule, \(\text{xO}\); it is included simply for symmetry. By contrast, the introduction rule, \(\text{xI}\), will be used extensively.

We are now in a position to write down the official formulation of indirect derivation of the first form (we discuss the second form in the next section).

**System Rule 7 (a show rule)**

**Indirect Derivation** (First Form)

If one has a show-line of the form ‘SHOW: \(\sim \text{A}\)’, then if one has \(\times\) as a later available line, and there are no subsequent uncancelled show-lines, then one is entitled to cancel ‘SHOW: \(\sim \text{A}\)’ and box off all subsequent lines. The annotation is ‘ID’.

**System Rule 8 (an assumption rule)**

If one has a show-line of the form ‘SHOW: \(\sim \text{A}\)’, then one is entitled to write down the un-negated formula \(\text{A}\) on the very next line, as an assumption. The annotation is ‘As’.

As with earlier rules, we offer a pictorial abbreviation of indirect derivation as follows.
With our new rules in hand, let us now go back and do our earlier derivation in accordance with the new rules.

**Example 1**

1. \(P \to Q\)  
2. \(P \to \neg Q\)  
3. \(\neg P\)  
4. \(P\)  
5. \(\neg P\)  
6. \(Q\)  
7. \(\neg Q\)  
8. \(\neg P\)  

On line (3), we are trying to show \(\neg P\), which is a negation, so we do it by ID. This entails writing down \(P\) on the next line as an assumption, and writing down `'SHOW: \neg P'` on the following line. On line (8), we obtain \(\neg P\) from lines (6) and (7), applying our new rule `\(\neg I\)`.

Let's do another simple example.

**Example 2**

1. \(P \to Q\)  
2. \(Q \to \neg P\)  
3. \(\neg P\)  
4. \(P\)  
5. \(\neg P\)  
6. \(Q\)  
7. \(\neg P\)  
8. \(\neg P\)  

In the previous two examples, \(\neg P\) is obtained from an atomic formula and its negation. Sometimes, \(\neg P\) comes from more complex formulas, as in the following examples.
Example 3

(1) \( \sim (P \lor Q) \) \hspace{1cm} \text{Pr}
(2) \text{SHOW: } \sim P \hspace{1cm} \text{ID}
(3) \text{P} \hspace{1cm} \text{As}
(4) \text{SHOW: } \times \hspace{1cm} \text{DD}
(5) \text{P} \lor Q \hspace{1cm} 3,\lor I
(6) \times \hspace{1cm} 1,5,\times I

Here, \( \times \) comes by \( \times I \) from \( P \lor Q \) and \( \sim (P \lor Q) \).

Example 4

(1) \( \sim (P \land Q) \) \hspace{1cm} \text{Pr}
(2) \text{SHOW: } P \rightarrow \sim Q \hspace{1cm} \text{CD}
(3) \text{P} \hspace{1cm} \text{As}
(4) \text{SHOW: } \sim Q \hspace{1cm} \text{ID}
(5) \text{Q} \hspace{1cm} \text{As}
(6) \text{SHOW: } \times \hspace{1cm} \text{DD}
(7) \text{P} \land Q \hspace{1cm} 3,5,\land I
(8) \times \hspace{1cm} 1,7,\times I

Here, \( \times \) comes, by \( \times I \), from \( P \land Q \) and \( \sim (P \land Q) \).

11. INDIRECT DERIVATION (SECOND FORM)

In addition to indirect derivation of the first form, we also add indirect derivation of the second form, which is very similar to the first form. Consider the following derivation problem.

(1) \( P \rightarrow Q \) \hspace{1cm} \text{Pr}
(2) \( \sim P \rightarrow Q \) \hspace{1cm} \text{Pr}
(3) \text{SHOW: } Q \hspace{1cm} ???

The same problem as before arises; we have no simple means of dealing with either premise. (3) is atomic, so we must show it by direct derivation, but that approach comes to a screeching halt!

Once again, let's do something sneaky (but completely legal!), and see where that leads.

(1) \( P \rightarrow Q \) \hspace{1cm} \text{Pr}
(2) \( \sim P \rightarrow Q \) \hspace{1cm} \text{Pr}
(3) \text{SHOW: } Q \hspace{1cm} ???
(4) \text{SHOW: } \sim \sim Q \hspace{1cm} ???

We have written down an additional show-line (which is completely legal, remember). The new problem facing us – to show \( \sim \sim Q \) – appears much more