Chapter 5: Derivations in Sentential Logic

Example 3

(1) \(\sim (P \lor Q)\) \hspace{1cm} \text{Pr}
(2) \(\text{SHOW: } \sim P\) \hspace{1cm} \text{ID}
(3) \(P\) \hspace{1cm} \text{As}
(4) \(\text{SHOW: } \times\) \hspace{1cm} \text{DD}
(5) \(P \lor Q\) \hspace{1cm} 3,\lor\text{I}
(6) \(\times\) \hspace{1cm} 1,5,\lor\text{I}

Here, \(\times\) comes by \(\lor\text{I}\) from \(P \lor Q\) and \(\sim (P \lor Q)\).

Example 4

(1) \(\sim (P \land Q)\) \hspace{1cm} \text{Pr}
(2) \(\sim P \land \sim Q\) \hspace{1cm} \text{CD}
(3) \(P\) \hspace{1cm} \text{As}
(4) \(\text{SHOW: } \sim Q\) \hspace{1cm} \text{ID}
(5) \(Q\) \hspace{1cm} \text{As}
(6) \(\text{SHOW: } \times\) \hspace{1cm} \text{DD}
(7) \(P \land Q\) \hspace{1cm} 3,5,\land\text{I}
(8) \(\times\) \hspace{1cm} 1,7,\land\text{I}

Here, \(\times\) comes, by \(\land\text{I}\), from \(P \land Q\) and \(\sim (P \land Q)\).

11. INDIRECT DERIVATION (SECOND FORM)

In addition to indirect derivation of the first form, we also add indirect derivation of the second form, which is very similar to the first form. Consider the following derivation problem.

(1) \(P \rightarrow Q\) \hspace{1cm} \text{Pr}
(2) \(\sim P \rightarrow Q\) \hspace{1cm} \text{Pr}
(3) \(\text{SHOW: } Q\) \hspace{1cm} ???

The same problem as before arises; we have no simple means of dealing with either premise. (3) is atomic, so we must show it by direct derivation, but that approach comes to a screeching halt!

Once again, let's do something sneaky (but completely legal!), and see where that leads.

(1) \(P \rightarrow Q\) \hspace{1cm} \text{Pr}
(2) \(\sim P \rightarrow Q\) \hspace{1cm} \text{Pr}
(3) \(\text{SHOW: } Q\) \hspace{1cm} ???
(4) \(\text{SHOW: } \sim \sim Q\) \hspace{1cm} ???

We have written down an additional show-line (which is completely legal, remember). The new problem facing us – to show \(\sim \sim Q\) – appears much more
promising; specifically, we are trying to show a negation, so we can attack it using indirect derivation, which yields the following part-derivation.

(1) \(P \rightarrow Q\)  
(2) \(\sim P \rightarrow Q\)  
(3) SHOW: Q  
(4) SHOW: \(\sim \sim Q\)  
(5) \(\sim Q\)  
(6) SHOW: \(\times\)  
(7) \(\sim P\)  
(8) \(\sim \sim P\)  
(9) \(\times\)  

The derivation is not complete. Line (3) is not cancelled. We are trying to show Q; we have in fact shown \(\sim \sim Q\). This is a near-hit because we can apply Double Negation to line (4) to get Q. This yields the following completed derivation.

(1) \(P \rightarrow Q\)  
(2) \(\sim P \rightarrow Q\)  
(3) SHOW: Q  
(4) SHOW: \(\sim \sim Q\)  
(5) \(\sim Q\)  
(6) SHOW: \(\times\)  
(7) \(\sim P\)  
(8) \(\sim \sim P\)  
(9) \(\times\)  
(10) \(Q\)  

This derivation presents something completely novel. Upon getting to line (9), we have shown \(\sim \sim Q\), which is marked by cancelling the ‘SHOW’ and boxing off the associated derivation. We can now use the formula \(\sim \sim Q\) in connection with the usual rules of inference. In this particular case, we apply double negation to obtain line (10). This is in accordance with the following principle.

As soon as one cancels a show-line ‘SHOW: \(A\)’, thus obtaining ‘SHOW: \(\sim A\)’, the formula \(A\) is available, at least until the show-line itself gets boxed off.

In order to abbreviate the above derivation somewhat, we enhance the method of indirect derivation so as to include, in effect, the above double negation maneuver. The intuitive formulation of this rule is given as follows.

**Indirect Derivation (Second Form)**

**Intuitive Formulation**

In order to show a formula \(A\), it is sufficient to show \(\times\), assuming its negation \(\sim A\).
As usual, the official formulation of the rule is more complex.

**System Rule 9 (a show rule)**

**Indirect Derivation (Second Form)**

If one has a show-line ‘SHOW: A’, then if one has X as a later available line, and there are no intervening un-cancelled show lines, then one is entitled to cancel ‘SHOW: A’ and box off all subsequent formulas. The annotation is ‘ID’

**System Rule 10 (an assumption rule)**

If one has a show-line ‘SHOW: A’, then one is entitled to write down the negation ~A on the very next line, as an assumption. The annotation is ‘As’

As usual, we also offer a pictorial version of the rule.

**Indirect Derivation (Second Form)**

With this new show-rule in hand, we can now rewrite our earlier derivation, as follows.

**Example 1**

(1) P → Q  
(2) ~P → Q  
(3) SHOW: Q  
(4) ~Q  
(5) SHOW: X  
(6) ~P  
(7) ~~P  
(8) X

In this particular problem, X is obtained by XI from ~P and ~~P.
Let's look at one more example of the second form of indirect derivation.

**Example 2**

(1) \(~(P \& \sim Q)\)  
(2) **SHOW**: \(P \rightarrow Q\)  
(3) \(P\)  
(4) **SHOW**: \(Q\)  
(5) \(\sim Q\)  
(6) **SHOW**: \(\checkmark\)  
(7) \(P \& \sim Q\)  
(8) \(\checkmark\)  

In this derivation we show \(P \rightarrow Q\) by conditional derivation, which means we assume \(P\) and show \(Q\). This is shown, in turn, by indirect derivation (second form), which means we assume \(\sim Q\) to show \(\checkmark\). In this particular problem, \(\checkmark\) is obtained by \(\checkmark I\) from \(P \& \sim Q\) and \(\sim(P \& \sim Q)\).

12. **SHOWING DISJUNCTIONS USING INDIRECT DERIVATION**

The second form of ID is very useful for showing atomic formulas, as demonstrated in the previous section. It is also useful for showing disjunctions. Consider the following derivation problem.

(1) \(\sim P \rightarrow Q\)  
(2) **SHOW**: \(P \lor Q\)  

We are asked to show a disjunction \(P \lor Q\). CD is not available because this formula is not a conditional. ID of the first form is not available because it is not a negation. DD is available but it does not work (except in conjunction with the double-negation maneuver). That leaves the second form of ID, which yields the following.

(1) \(\sim P \rightarrow Q\)  
(2) **SHOW**: \(P \lor Q\)  
(3) \(\sim(P \lor Q)\)  
(4) **SHOW**: \(\checkmark\)  
(5) ???

At this point, we are nearly stuck. We don't have the minor premise to deal with line (1), and we have no rule for dealing with line (3). So, what do we do? We can always write down a show-line of our own choosing, so we choose to write down ‘SHOW: \(\sim P\)’. This produces the following part-derivation.
(1) \( \sim P \rightarrow Q \)  
(2) SHOW: \( P \lor Q \)  
(3) \( \sim(P \lor Q) \)  
(4) SHOW: \( \times \)  
(5) SHOW: \( \sim P \)  
(6) \( P \)  
(7) SHOW: \( \times \)  
(8) \( P \lor Q \)  
(9) \( \times \)  
(10) ???

We are still not finished, but now we have shown \( \sim P \), so we can use it (while it is still available). This enables us to complete the derivation as follows.

(1) \( \sim P \rightarrow Q \)  
(2) SHOW: \( P \lor Q \)  
(3) \( \sim(P \lor Q) \)  
(4) SHOW: \( \times \)  
(5) SHOW: \( \sim P \)  
(6) \( P \)  
(7) SHOW: \( \times \)  
(8) \( P \lor Q \)  
(9) \( \times \)  
(10) \( Q \)  
(11) \( P \lor Q \)  
(12) \( \times \)

Lines 5-9 constitute a crucial, but completely routine, sub-derivation. Given how important, and yet how routine, this sub-derivation is, we now add a further inference-rule to our list. System SL is already complete as it stands, so we don't require this new rule. Adding it to system SL decreases its elegance. We add it purely for the sake of convenience.

The new rule is called tilde-wedge-out (\( \sim \lor O \)). As its name suggests, it is a rule for breaking down formulas that are negations of disjunctions. It is pictorially presented as follows.

\[
\text{Tilde-Wedge-Out (}\sim \lor O\text{)}
\]

\[
\begin{array}{c}
\sim(A \lor B) \\
\hline
\sim A & \sim(B)
\end{array}
\]

As with all inference rules, this rule applies exclusively to lines, not to parts of lines. In other words, the official formulation of the rule goes as follows.
Tilde-Wedge-Out ($\sim \lor O$)

If one has available a line of the form $\sim (\mathcal{A} \lor \mathcal{B})$, then one is entitled to write down both $\sim \mathcal{A}$ and $\sim \mathcal{B}$.

Once we have the new rule $\sim \lor O$, the above derivation is much, much simpler.

**Example 1**

1. $\sim P \rightarrow Q$  
2. $\text{SHOW: } P \lor Q$  
3. $\sim (P \lor Q)$  
4. $\text{SHOW: } \times$  
5. $\sim P$  
6. $\sim Q$  
7. $Q$  
8. $\times$

In the above problem, we show a disjunction using the second form of indirect derivation. This involves a general strategy for showing any disjunction, formulated as follows.

**General Strategy for Showing Disjunctions**

If you have a show-line of the form ‘SHOW: $A \lor B$’, then use indirect derivation: first assume $\sim [A \lor B]$, then write down ‘SHOW: $\times$’, then apply $\sim \lor O$ to obtain $\sim A$ and $\sim B$, then proceed from there.

In cartoon form:

```
SHOW: $A \lor B$  ID
$\sim [A \lor B]$  As
SHOW: $\times$
$\sim A$  $\sim \lor O$
$\sim B$  $\sim \lor O$
```

This particular strategy actually applies to any disjunction, simple or complex. In the previous example, the disjunction is simple (its disjuncts are
atomic). In the next example, the disjunction is complex (its disjuncts are not atomic).

**Example 2**

| (1) | (P ∨ Q) → (P & Q) | Pr |
| (2) | SHOW: (P & Q) ∨ (~P & ~Q) | ID |
| (3) | ~(P & Q) ∨ (~P & ~Q)] | As |
| (4) | SHOW: X | DD |
| (5) | ~(P & Q) | 3, ~v O |
| (6) | ~(~P & ~Q) | 3, ~v O |
| (7) | ~(P ∨ Q) | 1,5, → O |
| (8) | ~P | 7, ~v O |
| (9) | ~Q | 7, ~v O |
| (10) | ~P & ~Q | 8,9,& I |
| (11) | X | 6,10,X I |

The basic strategy is exactly like the previous problem. The only difference is that the formulas are more complex.

**13. FURTHER RULES**

In the previous section, we added the rule ~v O to our list of inference rules. Although it is not strictly required, it does make a number of derivations much easier. In the present section, for the sake of symmetry, we add corresponding rules for the remaining two-place connectives; specifically, we add ~& O, ~ → O, and ~↔ O. That way, we have a rule for handling any negated molecular formula.

Also, we add one more rule that is sometimes useful, the Rule of Repetition.

The additional negation rules are given as follows.

**Tilde-Ampersand-Out (~& O)**

\[
\neg (A \land B) \\
A \rightarrow \neg B
\]

**Tilde-Arrow-Out (~ → O)**

\[
\neg (A \rightarrow C) \\
A \land \neg C
\]