TRANSLATIONS IN MONADIC PREDICATE LOGIC

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1. INTRODUCTION

As we have noted in earlier chapters, the validity of an argument is a function of its form, as opposed to its specific content. On the other hand, as we have also noted, the form of a statement or an argument is not absolute, but rather depends upon the level of logical analysis we are pursuing.

We have already considered two levels of logical analysis – syllogistic logic, and sentential logic. Whereas syllogistic logic considers quantifier expressions (e.g., ‘all’, ‘some’) as the sole logical terms, sentential logic considers statement connectives (e.g., ‘and’, ‘or’) as the sole logical connectives. Thus, these branches of logic analyze logical form quite differently from one another.

Predicate logic subsumes both syllogistic logic and sentential logic; in particular, it considers both quantifier expressions and statement connectives as logical terms. It accordingly represents a deeper level of logical analysis. As a consequence of the deeper logical analysis, numerous arguments that are not valid, either relative to syllogistic logic, or relative to sentential logic, turn out to be valid relative to predicate logic. Consider the following argument.

\[(A) \quad \text{if at least one person will show up, then we will meet} \]
\[\quad \text{Adams will show up} \]
\[\quad / \text{we will meet} \]

First of all, argument (A) is not a syllogism, so it is not a valid syllogism. Next, if we symbolize (A) in sentential logic, we obtain something like the following.

\[(F) \quad P \rightarrow M \]
\[\quad A \]
\[\quad / M \]

Here ‘P’ stands for ‘at least one person will show up’, ‘A’ stands for ‘Adams will show up’, and ‘M’ stands for ‘we will meet’. It is easy to show (using truth tables) that (F) is not a valid sentential logic form.

Nevertheless, argument (A) is valid (intuitively, at least). What this means is that the formal techniques of sentential logic are not fully adequate to characterize the validity of arguments. In particular, (A) has further logical structure that is not captured by sentential logic. So, what we need is a further technique for uncovering the additional structure of (A) that reveals that it is indeed valid. This technique is provided by predicate logic.
2. **THE SUBJECT-PREDICATE FORM OF_ATOMIC_STATEMENTS**

Recall the distinction in sentential logic between the following sentences.

1. Jay and Kay are Sophomores
2. Jay and Kay are roommates

Whereas the former is equivalent to a conjunction, namely,

1* Jay is a Sophomore and Kay is a Sophomore,

the latter is an atomic statement, having no structure from the viewpoint of sentential logic. In particular, whereas in (1) ‘and’ is used *conjunctively* to assert something about Jay and Kay individually, in (2) ‘and’ is used *relationally* – to assert that a certain relation holds between Jay and Kay.

In predicate logic, we are able to uncover the additional logical structure of (2); indeed, we are able to uncover the additional logical structure of (1) as well. In particular, we are able to display atomic formulas as consisting of a predicate and one or more subjects.

Consider the atomic statements that compose (1).

3. Jay is a Sophomore
4. Kay is a Sophomore

Each of these consists of two grammatical components: a *subject* and a *predicate*. In (3), the subject is ‘Jay’, and the predicate is ‘...is a sophomore’; in (4), the subject is ‘Kay’, and the predicate is the same, ‘...is a sophomore’.

Next, consider the sole atomic statement in (2), which is (2) itself.

5. Jay and Kay are roommates

This may be paraphrased as follows.

5* Jay is a roommate of Kay

Unlike (3) and (4), this sentence has two grammatical subjects – ‘Jay’ and ‘Kay’. In addition to the subjects, there is also a predicate – ‘...is a roommate of...’

The basic idea in the three examples so far is that an atomic sentence can be grammatically analyzed into a predicate and one or more subjects. In order to further emphasize this point, let us consider a slightly more complicated example, involving several subjects in addition to a single predicate.

6. Chris is sitting between Jay and Kay

Once more ‘and’ is used relationally rather than conjunctively; in particular (6) is not a conjunction, but is rather atomic. In this case, the predicate is fairly complex:
...is sitting between...and..., and there are three grammatical subjects:

Chris, Jay, Kay

We now state the first principle of predicate logic.

In predicate logic, every atomic sentence consists of one *predicate* and one or more *subjects*.

### 3. PREDICATES

Every predicate has a degree, which is a number. If a predicate has degree one, we call it a *one-place predicate*; if it has degree two, we call it a *two-place predicate*; and so forth.

In principle, for every number $n$, there are predicates of degree $n$, (i.e., $n$-place predicates). However, we are going to concentrate primarily on $1$-place, $2$-place, and $3$-place predicates, in that order of emphasis.

To say that a predicate is a one-place predicate is to say that it takes a single grammatical subject. In other words, a one-place predicate forms a statement when combined with a *single* subject. The following are examples.

___ is clever
___ is a Sophomore
___ sleeps soundly
___ is very unhappy

Each of these is a $1$-place predicate, because it takes a single term to form a statement; thus, for example, we obtain the following statements.

Jay is clever  
Kay is a Sophomore  
Chris sleeps soundly  
Max is very unhappy

On the other hand, a *two-place predicate* takes two grammatical subjects, which is to say that it forms a statement when combined with two names. The following are examples.
___ is taller than ___
___ is south of ___
___ admires ___
___ respects ___
___ is a cousin of ___

Thus, for example, using various pairs of individual names, we obtain the following statements.

Jones is taller than Smith
New York is south of Boston
Jay admires Kay
Kay respects Jay
Jay is a cousin of Kay

Finally, a three-place predicate takes three grammatical subjects, which is to say that it forms a statement when combined with three names. The following are examples.

___ is between ___ and ___
___ is a child of ___ and ___
___ is the sum of ___ and ___
___ borrowed ___ from ___
___ loaned ___ to ___
___ recommended ___ to ___

Thus, for example, we may obtain the following statements from these predicates.

New York is between Boston and Philadelphia
Chris is a child of Jay and Kay
11 is the sum of 4 and 7
Jay borrowed this pen from Kay
Kay loaned this pen to Jay
Kay recommended the movie “Casablanca” to Jay

One-place predicates (also called monadic predicates) may be thought of as denoting properties (e.g., the property of being tall), whereas multi-place (1,2,3-place) predicates (also called polyadic predicates) may be thought of as denoting relations (e.g., the relation between two things when one is taller than the other).

Sometimes, the study of predicate logic is formally divided into monadic predicate logic (also called property logic) and polyadic predicate logic (also called relational logic). In this text, we do not formally divide the subject in this way. On the other hand, we deal primarily with monadic predicate logic in the present chapter, leaving polyadic predicate logic for the next chapter.
4. **SINGULAR TERMS**

Predicate logic analyzes every atomic sentence into a predicate and one or more subjects. In the present section, we examine the latter in a little more detail. In the previous section, the alert reader probably noticed that diverse sorts of expressions were substituted into the blanks of the predicates. Not only did we use names of people, but we also used numerals (which are names of numbers), the name of a movie, and even a demonstrative noun phrase ‘this pen’.

These are all examples of *singular terms* (also called individual terms), which include four sorts of expressions, among others.

1. proper nouns
2. definite descriptions
3. demonstrative noun phrases
4. pronouns

Examples of proper nouns include the following.

Jay, Kay, Chris, etc.
George Washington, John F. Kennedy, etc.
Paris, London, New York, etc.
Jupiter, Mars, Venus, etc.
1, 2, 3, 23, 45, etc.

Examples of definite descriptions that are singular terms include the following.

the largest river in the world
James Joyce's last book
the president of the U.S.
the square root of 2
the first person to finish the exam

Examples of demonstrative noun phrases that are singular terms include the following.

the person over there
this person, this pen, etc.
that person, that pen, etc.

The use of demonstrative noun phrases generally involves *pointing*, either explicitly or implicitly.

Examples of pronouns that are singular terms are basically all the third person singular pronouns,

he, she, it, him, her,

as well as “wh” expressions such as

who, whom, which (that), what, when, where, why.
Having seen various examples of singular terms, it is equally important to see examples of noun-like expressions that do not qualify as singular terms. These might be called, by analogy, *plural terms*.

**Examples of Plural Terms**

- the people who play for the New York Yankees
- the five smartest persons in the class
- James Joyce's books
- the European cities
- the natural numbers
- the people standing over there
- they, them, these, those

Note carefully that many people use ‘they’ and ‘them’ as singular pronouns. Consider the following example.

(1) I have a date tonight with a music major; I am meeting *them* at the concert hall.

One's response to hearing the word ‘them’ should be “exactly how many people do you have a date with?” or “is your date a schizophrenic?” More than likely, your date is a man, in which case your date is a "him", or is a woman, in which case your date is a "her". Unless your date consists of several people, it is not a "them".

Another very common example in which ‘they/them/their’ is used (incorrectly) as a singular pronoun is the following.

(2) Everyone in the class likes their roommate.

In times long past, literate people thought that ‘he’, ‘him’, and ‘his’ had a use as singular third person *neutral* pronouns. In those care-free times, when men were men (and so were women!), the grammatically correct formulation of (2) would have been the following.

(*2) Everyone in the class likes his roommate.

Nowadays, in the U.S. at least, many literate people reject the neutrality of ‘he’, ‘him’, and ‘his’ and accordingly insist on rewriting the above sentence in the following (slightly stilted) manner.

(!2) Everyone in the class likes his or her roommate.

Notwithstanding the fact that illiterate people use ‘they’, ‘them’, and ‘their’ as singular pronouns, these words are in fact *plural* pronouns, as can quickly be seen by examining the following two sentences.

(1) they are tall (plural verb form)
(2) they is tall (singular verb form)
A singular term refers to a single individual – a person, place, thing, event, etc., although perhaps a complex one, like IBM, or a very complex one, like the Renaissance. In order to decide whether a noun phrase qualifies as a singular term, the simplest thing to do is to check whether the noun phrase can be used properly with the singular verb form ‘is’. If the noun phrase requires the plural form ‘are’, then it is not a singular term, but is rather a plural term.

Let us conclude by stating a further, very important, principle of the grammar of predicate logic.

In predicate logic, every subject is a singular term.

5. ATOMIC FORMULAS

Having discussed the manner in which every atomic sentence of predicate logic is decomposed into a predicate and (singular) subject(s), we now introduce the symbolic apparatus by which the form of such a sentence is formally displayed.

In sentential logic, you will recall, atomic sentences are abbreviated by upper case letters of the Roman alphabet. The fact that they are symbolized by letters reflects the fact that they are regarded as having no further logical structure. By contrast, in predicate logic, every atomic sentence is analyzed into its constituents, being its predicate and its subject or subjects.

In order to distinguish these constituents, we adopt a particular notational convention, which is simple if not entirely intuitive. This convention is presented as follows.

(1) Predicates are symbolized by upper case letters.
(2) Singular terms are symbolized by lower case letters.
(3) Every atomic sentence is symbolized by juxtaposing the associated subject and predicate letters.
(4) In particular, in each atomic sentence, the predicate letter goes first and is followed by the subject letter(s).
The following are examples.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Predicates:</strong></td>
<td></td>
</tr>
<tr>
<td>___ is tall</td>
<td>T</td>
</tr>
<tr>
<td>___ is a Freshman</td>
<td>F</td>
</tr>
<tr>
<td>___ respects____</td>
<td>R</td>
</tr>
<tr>
<td>___ is a cousin of___</td>
<td>C</td>
</tr>
<tr>
<td>___ is between ___ and ___</td>
<td>B</td>
</tr>
<tr>
<td><strong>Singular Terms:</strong></td>
<td></td>
</tr>
<tr>
<td>Jay</td>
<td>j</td>
</tr>
<tr>
<td>Kay</td>
<td>k</td>
</tr>
<tr>
<td>New York City</td>
<td>n</td>
</tr>
<tr>
<td>Jupiter</td>
<td>j</td>
</tr>
<tr>
<td>the tallest person in class</td>
<td>t</td>
</tr>
<tr>
<td>the movie “Casablanca”</td>
<td>c</td>
</tr>
<tr>
<td><strong>Sentences:</strong></td>
<td></td>
</tr>
<tr>
<td>Jay is tall</td>
<td>Tj</td>
</tr>
<tr>
<td>Kay is a Freshman</td>
<td>Fk</td>
</tr>
<tr>
<td>Jay respects Kay</td>
<td>Rjk</td>
</tr>
<tr>
<td>Kay is a cousin of Jay</td>
<td>Ckj</td>
</tr>
<tr>
<td>Chris is between Jay and Kay</td>
<td>Bcjk</td>
</tr>
</tbody>
</table>

From occasion to occasion, different predicates can be abbreviated by the same letter; likewise for singular terms. However, in any given context (a statement or argument), one must be careful to use different letters to abbreviate different names. The letter ‘j’ can stand for ‘Jay’ or for ‘Jupiter’, but if ‘Jay’ and ‘Jupiter’ appear in the same statement or argument, then we cannot use ‘j’ to abbreviate both of them; for example, we might use ‘j’ for ‘Jay’ and ‘u’ for ‘Jupiter’.

Notice that we use lower case letters to abbreviate all singular terms, including definite descriptions. Unlike proper nouns, definite descriptions have further logical structure, and this further structure is revealed and examined in more advanced branches of logic. However, for the purposes of intro logic, definite descriptions have no further logical structure; they are simply singular terms, and are accordingly abbreviated simply by lower case letters.
6. VARIABLES AND PRONOUNS

So far we have concentrated on singular terms that might be called constants. In addition to constants there are also variables. Variables play the same role in predicate logic that (singular third-person) pronouns play in ordinary language; specifically, they are used for cross-referencing inside a sentence or larger linguistic unit. Furthermore, variables play the same role in predicate logic that variables play in symbolic arithmetic (called algebra in high school); specifically, they enable us to refer to individuals (e.g., individual numbers), without referring to any particular individual (number). This is very useful, as we shall see shortly, in making general claims.

Concerning symbolization, whereas we use the lower case letters ‘a’, ‘b’, ‘c’, ..., ‘w’ as constants, we use the remaining lower case letters ‘x’, ‘y’, ‘z’ as variables. If it turns out that we need more than 26 constants or variables, then we will subscript these with numerals to obtain, for example, ‘a₂’, ‘y₃’, ‘z₅₀’, etc. Thus, in principle, there are infinitely many constants and variables.

In order to avoid using subscripted variables, we also reserve the right to "requisition" constants to use as variables, if the need should arise. So, for example, if we need six variables, but only a few constants, then we will "draft" ‘u’, ‘v’, and ‘w’ into service as variables. If this should happen, it will be explicitly announced. For the most part, however, in intro logic we need only three variables, and there is no need to recruit constants.

When we combine a predicate with one or more singular terms, we obtain a formula of predicate logic. When one or more of these singular terms is a variable, we obtain an open formula. Open formulas of predicate logic correspond to open sentences of natural language.

Consider the following sentences of arithmetic.

(1) 2 is even  \[ Et \]
(2) 3 is larger than 4  \[ Lt \]
(3) it is even  \[ Ex \]
(4) this is larger than that  \[ Lxy \]

Whereas (1) and (2) are closed sentences, and their symbolizations, to the right, are closed formulas, (3) and (4) are open sentences, and their symbolizations are open formulas.

So, what is the difference between open and closed sentences, anyway? The difference can be described by saying that, whereas (1) and (2) express propositions and are accordingly true or false, (3) and (4) do not (by themselves) express propositions and are accordingly neither true nor false.

On the other hand (this is the tricky part!), even though it does not autonomously express a proposition, an open sentence can be used to assert a proposition – specifically, by uttering it while "pointing" at a particular object or objects. If we "point" at the number two (insofar as that is possible), and say
“it/this/that is even”, then we have asserted the proposition that the number two is even; indeed, we have asserted a true proposition. Similarly, when we successively point at the number two and the number five, and say “this is larger than that”, then we have asserted the proposition that two is larger than five; we have asserted a proposition, but a false proposition.

A closed sentence, by contrast, can be used to assert a proposition, even without having to point. If I say “two is even”, I need not point at the number two in order to assert a proposition; the sentence does it for me.

One way to describe the difference between open and closed sentences is to say that, unlike closed sentences, open sentences are essentially *indexical* in character, which is to say that their use essentially involves *pointing*. (Here, think of the *index* finger, as used for pointing.) This pointing can be fairly straightforward, but it can also be oblique and subtle. This pointing can also be either *external* or *internal* to the sentence in which the indexical (i.e., pointing) expression occurs.

For example, in the sentence about the date with the music major, the pronoun refers to (points at) something external; the ‘he or she’ refers to the particular person about whom the speaker is talking. By contrast, in the sentence about roommates, the ‘his or her’ refers, not externally to a particular person, but rather internally to the expression ‘everyone’.

Another use of internal pointing involves the following indexical expressions.

(1) the former
(2) the latter
(3) the party of the first part
(4) the party of the second part

The latter two expressions (an example of pointing!) are used almost exclusively in legal documents, and we will not examine them any further. The former two expressions, on the other hand, are important expressions in logic. If I refer to a music major and a business major, in that order, then if I say “the former respects the latter”, I am saying that the music major respects the business major. If I say instead “he respects her”, then it is not clear who respects whom. Thus, the words ‘former’ and ‘latter’ are useful substitutes for ordinary pronouns.

We conclude this section by announcing yet another principle of the grammar of predicate logic.

In an atomic formula, every subject is either a *constant* or a *variable*.
7. COMPOUND FORMULAS

We have now described the atomic formulas of predicate logic; every such formula consists of an n-place predicate letter followed by n singular terms, each one being either a constant or a variable. The atomic formulas of predicate logic play exactly the same role that atomic formulas play in sentential logic; in particular, they can be combined with connectives to form molecular formulas.

We already know how to construct molecular formulas from atomic formulas in sentential logic. This skill carries over directly to predicate logic, the rules being precisely the same. If we have a formula, we can form its negation; if we have two formulas, we can form their conjunction, disjunction, conditional, and biconditional. The only difference is that the simple statements we begin with are not simply letters, as in sentential logic, but are rather combinations of predicate letters and singular terms.

The following are examples of compound statements in predicate logic, followed by their symbolizations.

(1) if Jay is a Freshman, then Kay is a Freshman \( F_j \rightarrow F_k \)
(2) Kay is not a Freshman \( \sim F_k \)
(3) neither Jay nor Kay is a Freshman \( \sim F_j \land \sim F_k \)
(4) Jay respects Kay, but Kay does not respect Jay \( R_{jk} \land \sim R_{kj} \)

Next, we note that either (or both) of the proper nouns ‘Jay’ and ‘Kay’ can be replaced by pronouns. Correspondingly, either (or both) of the constants ‘j’ and ‘k’ can be replaced by variables (for example, ‘x’ and ‘y’). We accordingly obtain various open sentences (formulas). For example, taking (1), we can construct the following open statements and associated open formulas.

(1) if Jay is a Freshman, then Kay is a Freshman \( F_j \rightarrow F_k \)
(1a) if Jay is a Freshman, then she is a Freshman \( F_j \rightarrow F_y \)
(1b) if he is a Freshman, then Kay is a Freshman \( F_x \rightarrow F_k \)
(1c) if he is a Freshman, then she is a Freshman \( F_x \rightarrow F_y \)

8. QUANTIFIERS

We have already seen that compound formulas can be constructed using the connectives of sentential logic. In addition to these truth-functional connectives, predicate logic has additional compound forming expressions – namely, the quantifiers.

Quantifiers are linguistic expressions denoting quantity in some form. Examples of quantifiers in English include the following.
every, all, each, both, any, either
some, most, many, several, a few
none, neither
at least one, at least two, etc.
at most one, at most two, etc.
exactly one, exactly two, etc.

These expressions are typically combined with noun phrases to produce sentences, such as the following.

every Freshman is clever
at least one Sophomore is clever
no Senior is clever
many Sophomores are clever
several Juniors are clever

In addition to these quantifier expressions, there are also derivative expressions, contractions, involving ‘thing’ and ‘one’.

everyone, everything, someone, something, no one, nothing

These yield sentences such as the following

everyone is clever
everything is clever
someone is clever
something is clever
no one is clever
nothing is clever

Recall that there are numerous statement connectives in English, but in sentential logic we concentrate on just a few, logically fruitful, ones. Similarly, even though there are numerous quantifier expressions in English, in predicate logic we concentrate only on a couple of them, given as follows.

every
at least one

Not only do we concentrate on these two quantifier concepts, we render them very general, as follows.

everything is such that...
there is at least one thing such that...

Although these expressions are somewhat stilted (much like the official expression for negation ‘it is not true that...’), they are sufficiently general to be used in a much wider variety of contexts than more colloquial quantifier expressions.

If this is not stilted enough, we must add one further feature to the above quantifiers, in order to obtain the official quantifiers of predicate logic. Recall
that a pronoun can point internally, and in particular, it can point at a quantifier expression in the sentence. In the sentence

everyone likes his/her roommate

the pronoun ‘his/her’ points at the quantifier ‘everyone’. But what if the sentence in question has more than one quantifier? Consider the following.

everyone knows someone who respects his/her mother

This sentence is ambiguous, because it isn’t clear what the pronoun ‘his/her’ points at. This sentence might be paraphrased in either of the following ways.

everyone knows someone who respects the former’s mother
everyone knows someone who respects the latter’s mother

The additional feature needed by the quantifiers above is an index, in order to allow clear and consistent cross-referencing inside of sentences in which they appear. Since we are using variables as pronouns, it is convenient to use the very same symbolic devices as quantifier indices as well.

Thus, every quantifier comes with an index (a variable) attached to it. We thus obtain the following quantifier expressions.

everything x is such that...
everything y is such that...
everything z is such that...

there is at least one thing x such that...
there is at least one thing y such that...
there is at least one thing z such that...

These are symbolized respectively as follows.

\[
\forall x \, \forall y \, \forall z \\
\exists x \, \exists y \, \exists z
\]

Historically, the upside-down ‘A’ derives from the word ‘all’, and the backwards ‘E’ derives from the word ‘exist’. Whereas the expressions ‘\( \forall x \)’, ‘\( \forall y \)’, ‘\( \forall z \)’ are called universal quantifiers, the expressions ‘\( \exists x \)’, ‘\( \exists y \)’, ‘\( \exists z \)’ are called existential quantifiers.

For every variable, there are two quantifiers, a universal quantifier, and an existential quantifier. Grammatically, a quantifier is a one-place connective, just like negation ~. In other words, we have the following grammatical principle.

If \( F \) is a formula, then so are all the following.

\[
\forall x F, \, \forall y F, \, \forall z F \\
\exists x F, \, \exists y F, \, \exists z F
\]
Of course, in forming the compound formula, the outer parentheses (if any) of the formula $F$ must be restored before prefixing the quantifier. This is just like negation. We will see examples of this later.

We now have the official quantifier expressions of predicate logic. How do they combine with other formulas to make quantified formulas? The basic idea (but not the whole story) is that one begins with an open formula involving (say) the variable ‘$x$’, and one prefixes ‘$\forall x$’ to obtain a universally quantified formula, or one prefixes ‘$\exists x$’ to obtain an existentially quantified formula.

For example, we can begin with the following open formula,

$$Fx: \text{it is fascinating}$$

and prefix either ‘$\forall x$’ or ‘$\exists x$’ to obtain the following formulas.

- $\forall xFx$: everything $[x]$ is such that it $[x]$ is fascinating
- $\exists xFx$: there is at least one thing $[x]$ such that it $[x]$ is fascinating

In each case, I have divided the sentence into a quantifier and an open formula. The variables are placed in parentheses, since they are not really part of the English sentence; rather, they are used to cross-reference the pronoun ‘it’. In particular, the fact that ‘$x$’ is used for both the quantifier and the pronoun indicates that ‘it’ points back at (cross-references) the quantifier expression.

This is the simplest case, one in which the open formula $A$ is atomic. It can also be molecular; it can even be a quantified formula (a great deal more about this in the next chapter). The following are all examples of open formulas involving ‘$x$’ together with the resulting quantified formulas. Notice the appearance of the parentheses in (2) and (3).

<table>
<thead>
<tr>
<th>Open Formula:</th>
<th>Universal Formula:</th>
<th>Existential Formula:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sim Fx$</td>
<td>$\forall x \sim Fx$</td>
<td>$\exists x \sim Fx$</td>
</tr>
<tr>
<td>$Fx &amp; Gx$</td>
<td>$\forall x (Fx &amp; Gx)$</td>
<td>$\exists x (Fx &amp; Gx)$</td>
</tr>
<tr>
<td>$Fx \rightarrow Gx$</td>
<td>$\forall x (Fx \rightarrow Gx)$</td>
<td>$\exists x (Fx \rightarrow Gx)$</td>
</tr>
<tr>
<td>$Rxy$</td>
<td>$\forall x Rxy$</td>
<td>$\exists x Rxy$</td>
</tr>
<tr>
<td>$\exists y Rxy$</td>
<td>$\forall x \exists y Rxy$</td>
<td>$\exists x \exists y Rxy$</td>
</tr>
</tbody>
</table>

The pairs to the right are all examples of quantified formulas, universal formulas and existential formulas respectively. These can in turn be combined using any of the sentential logic connectives, to obtain (e.g.) the following compound formulas.
<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>(\forall x \sim Fx \lor \forall x(Fx &amp; Gx))</td>
<td>disjunction</td>
</tr>
<tr>
<td>7</td>
<td>(\sim \forall xRxj; \sim \exists xRxj; \sim \forall x \exists yRxy; \sim \exists x \exists yRxy)</td>
<td>negations</td>
</tr>
<tr>
<td>8</td>
<td>(\forall xRxj \rightarrow \forall x \exists yRxy; \exists xRxj \rightarrow \exists x \exists yRxy)</td>
<td>conditionals</td>
</tr>
</tbody>
</table>

At this stage, the important thing is not necessarily to be able to read the above formulas, but to be able to recognize them as formulas. Toward this end, keep in clear sight the rules of formula formation in predicate logic, which are sketched as follows.

**Definition of Formula in Predicate Logic:**

**Atomic Formulas:**

1. If \(P\) is a predicate letter of degree \(n\), then \(P\) followed by \(n\) singular terms is an atomic formula.
2. Nothing else is an atomic formula.

**Formulas:**

1. Every atomic formula is a formula.
2. If \(\mathcal{A}\) is a formula, then so is \(\sim \mathcal{A}\).
3. If \(\mathcal{A}\) and \(\mathcal{B}\) are formulas then so are the following.
   - \((\mathcal{A} \& \mathcal{B})\)
   - \((\mathcal{A} \lor \mathcal{B})\)
   - \((\mathcal{A} \rightarrow \mathcal{B})\)
   - \((\mathcal{A} \leftrightarrow \mathcal{B})\)
4. If \(\mathcal{A}\) is a formula, then so are the following.
   - \(\forall x \mathcal{A}, \forall y \mathcal{A}, \forall z \mathcal{A},\) etc.
   - \(\exists x \mathcal{A}, \exists y \mathcal{A}, \exists z \mathcal{A},\) etc.
5. Nothing else is a formula.

**9. COMBINING QUANTIFIERS WITH NEGATION**

As noted at the end of the previous section, any formula can be prefixed by either a universal quantifier or an existential quantifier, just as any formula can be prefixed by negation, and the result is another formula.

In the present section, we concentrate on the way in which negation interacts with quantifiers.

Let us start with the following open formula.

1. \(Px\) it is perfect

Then let us quantify it both universally and existentially, as follows.
(2) $\forall x P x$ everything is such that it is perfect

(3) $\exists x P x$ at least one thing is such that it is perfect

These can in turn be negated, yielding the following formulas.

(4) $\neg \forall x P x$ it is not true that everything is such that it is perfect

(5) $\neg \exists x P x$ it is not true that at least one thing is such that it is perfect

Before considering more colloquial paraphrases of the above sentences, let us consider an alternative tack. Let us first negate ‘P x’ to obtain the following.

(6) $\neg P x$ it is not true that it is perfect

The latter sentence may be paraphrased as either of the following.

it is not perfect
It is imperfect

Many adjectives have ready-made negations (happy/unhappy, friendly/unfriendly, possible/impossible); most adjectives, however, do not have natural negations. On the other hand, we can always produce the negation of any adjective simply by prefixing ‘non-’ in front of the adjective.

Now, let us take the negated formula ‘$\neg P x$’ and quantify in the two ways, which yields the following.

(7) $\forall x \neg P x$ everything is such that it is not true that it is perfect

everything is such that it is not perfect

everything is such that it is imperfect

(8) $\exists x \neg P x$ at least one thing is such that it is not true that it is perfect

at least one thing is such that it is not perfect
at least one thing is such that
it is imperfect

Having written down all the simple formulas involving negation and quantifiers, let us now consider the idiomatic rendering of these sentences. First, to say
everything is such that it is perfect

is equivalent to saying
everything has a certain property – it is perfect.

These two sentences are simply verbose ways of saying
everything is perfect.

Similarly, to say
at least one thing is such that it is perfect,

which is an alternative to
there is at least one thing such that it is perfect,

is equivalent to saying
at least one thing has a certain property – it is perfect.

These two sentences are simply verbose ways of saying
at least one thing is perfect.

The latter sentence, in turn, can be thought of as one way of rendering precise the following.
something is perfect

Along similar lines, recall the way that the negation operator works; the official form of negation involves prefixing ‘it is not true that’ in front of the sentence in question. Thus, for example, one obtains the following.
it is not true that it is perfect

Recall that this is equivalent to the following more colloquial expression.
it is not perfect

The advantage of the verbose forms of negation and quantification is grammatical generality; we can always produce the official negation or quantification of a sentence, but we cannot always easily produce the colloquial negation or quantification.

For example, consider the following.
everything is such that
  it is not true that
    it is perfect,

which is equivalent to

  everything is such that
    it is not perfect.

Following the above line of reasoning concerning colloquial quantification, the natural paraphrase of this is the following.

  everything is not perfect

Unfortunately, the placement of ‘not’ in this sentence makes it unclear whether it modifies ‘is’ or ‘perfect’; accordingly, this sentence is ambiguous in meaning between the following pair of sentences.

  everything isn't perfect
    (i.e., not everything is perfect)

  everything is non-perfect

These are not equivalent; if, some things are perfect and some things are not, the first is true, but the second is false.

The original sentence,

  everything is such that it is not perfect,

says that everything has the property of being non-perfect (imperfect), or

  everything is non-perfect (imperfect).

To say that everything is non-perfect (imperfect) is equivalent to saying

  nothing is perfect,

which is much stronger than

  not everything is perfect.

The latter sentence is a colloquial paraphrase of

  it is not true that everything is perfect,

which is a colloquial paraphrase of

  it is not true that
    everything is such that
      it is perfect.

This is precisely formula (4) above.
Now, if not everything is perfect, then there is at least one thing that isn’t perfect, and conversely. To say the latter, we write

at least one thing is such that it is not perfect,

which is formula (8) above.

Finally, consider formula (5)

\[ \neg \exists x \neg Px \quad \text{it is not true that} \]
\[ \quad \text{at least one thing is such that} \]
\[ \quad \text{it is perfect} \]

which is equivalent to

\[ \text{it is not true that} \]
\[ \text{at least one thing is perfect.} \]

The number of things that are perfect is either zero, one, two, three, etc. To say that at least one thing is perfect is to say that the number of perfect things is at least one, that is, the number is not zero. To say that this is not true is to say that the number of perfect things is zero, which is to say

nothing is perfect.

Thus, we basically have six colloquial sentences.

(c1) everything is perfect
(c2) something is perfect (i.e., at least one thing is perfect)
(c3) everything is imperfect
(c4) something is imperfect
(c5) not everything is perfect
(c6) nothing is perfect

These correspond to the following formulas of predicate logic.

(f1) \( \forall x Px \)
(f2) \( \exists x Px \)
(f3) \( \forall x \neg Px \)
(f4) \( \exists x \neg Px \)
(f5) \( \neg \forall x Px \)
(f6) \( \neg \exists x Px \)

As noted earlier, two pairs of formulas are equivalent. In particular:

\( \forall x \neg Px \quad \text{[everything is imperfect]} \)

is equivalent to

\( \neg \exists x Px \quad \text{[nothing is perfect]}, \)

and
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\[ \exists x \sim Px \quad \text{[something is imperfect]} \]
is equivalent to

\[ \sim \forall xPx \quad \text{[not everything is perfect]} \]

These are instances of two very general equivalences, which may be stated as follows.

\[ \sim \forall x = \exists x \sim \]
\[ \sim \exists x = \forall x \sim \]

What this means is that for any formula \( A \), however complex, we have the following.

- \( \sim \forall x A \) is equivalent to \( \exists x \sim A \).
- \( \sim \exists x A \) is equivalent to \( \forall x \sim A \).

In order to understand them better, it might be worthwhile to compare these two equivalences with their counterparts in sentential logic – deMorgan's laws. In their simplest form, these laws of logic are stated as follows.

\[(dM1) \quad \sim(A \& B) \text{ is equivalent to } \sim A \sim B.\]
\[(dM2) \quad \sim(A \sim B) \text{ is equivalent to } \sim A \& \sim B.\]

But there are more general forms as well, given as follows.

\[(M1) \quad \sim(A_1 \& A_2 \& \ldots \& A_n) \text{ is equivalent to } A_1 \sim A_2 \sim \ldots \sim A_n\]
\[(M2) \quad \sim(A_1 \sim A_2 \sim \ldots \sim A_n) \text{ is equivalent to } \sim A_1 \& \sim A_2 \& \ldots \& \sim A_n\]

In other words, the negation of any conjunction, however long, is equivalent to a corresponding disjunction of negations, and similarly, the negation of any disjunction, however long, is equivalent to a corresponding conjunction of negations.

But what does this have to do with universal and existential quantifiers. Well, imagine for a moment there are exactly two things in the universe – call them a and b, respectively. In such a universe, which is very small, every universally quantified statement is equivalent to a conjunction, and every existentially quantified statement is equivalent to a disjunction. In particular, we have the following.

- everything is F :: a is F, and b is F
- something is F :: a is F, and/or b is F

Or, in formulas:

\[ \forall xFx :: F_a & F_b \]
Similarly, if there are exactly three things in the universe (a, b, c), then we have the following equivalences.

everything is F :: a is F, and b is F, and c is F
something is F :: a is F, and/or b is F, and/or c is F

Or, in formulas:
\[ \forall x Fx \equiv Fa \& Fb \& Fc \]
\[ \exists x Fx \equiv Fa \lor Fb \lor Fc \]

This can be generalized to any (finite) number of things in the universe; for every universally/ existentially quantified statement, there is a corresponding conjunction/ disjunction of suitable length.

Having seen what the equivalence looks like in general, let us concentrate on the simplest non-trivial version – a universe with just two things (a and b) in it.

Next, let us consider what happens when we combine quantifiers with negation? First, the simplest.

everything is not-F :: a is not F and b is not F
something is not-F :: a is not F and/or b is not F

Or, in formulas:
\[ \forall x \sim Fx \equiv \sim Fa \& \sim Fb \]
\[ \exists x \sim Fx \equiv \sim Fa \lor \sim Fb \]

Negating the quantified statements yields:

not everything is F :: not(a is F and b is F)
nothing is F :: not something is F :: not(a is F and/or b is F)

Or, in formulas:
\[ \sim \forall x Fx \equiv \sim (Fa \& Fb) \]
\[ \sim \exists x Fx \equiv \sim (Fa \lor Fb) \]

Finally, we obtain the following chain of equivalences.

\[ \sim \forall x Fx \equiv \sim (Fa \& Fb) :: \sim Fa \lor \sim Fb :: \exists x \sim Fx \]
\[ \sim \exists x Fx \equiv \sim (Fa \lor Fb) :: \sim Fa \& \sim Fb :: \forall x \sim Fx \]

The same procedure can be carried out with three, or four, or any number of, individuals.
Note: In the previous example, the formula $\mathfrak{A}$ is simple, being $Fx$. In general, $\mathfrak{A}$ may be complex – for example, it might be the formula $(Fx \rightarrow Gx)$. Then $\sim \mathfrak{A}$ is the negation of the entire formula, which is $\sim(Fx \rightarrow Gx)$. (Notice that the parentheses are optional in the conditional, but not in its negation.)

10. SYMBOLIZING THE STATEMENT FORMS OF SYLLOGISTIC LOGIC

Recall that the statement forms of syllogistic logic are given as follows.

(f1) all A are B
(f2) some A are B
(f3) no A are B
(f4) some A are not B

These are all stated in the plural form. In order to translate these into predicate logic, the first thing we must do is to convert each plural form into the corresponding closest singular form.

(s1) every A is B  [every A is a B]
(s2) some A is B  [some A is a B]
(s3) no A is B  [no A is a B]
(s4) some A is not B  [some A is not a B]

Examples of sentences in these forms are given as follows.

(e1) every astronaut is brave
(e2) some astronaut is brave
(e3) no astronaut is brave
(e4) some astronaut is not brave

Note that the simple predicate ‘is brave’ can be replaced by the longer expression ‘is a brave person’.

The next thing we must do is to convert the specific quantifier expressions ‘every/some/no A’ into the corresponding expressions involving general quantifiers ‘every/some/thing is such that...’

Consider (s1); to say

every A is a B

is to say

everything that is A is B,
or if we have persons exclusively in mind,
everyone who is A is B.

For example, we could read the latter as follows.
everyone who is an astronaut is brave

We know how to formalize ‘everything (everyone) is B’.

\[ \forall x \text{Bx} \]

But we don’t want to say that everything is B, just every A is B. How do we add the clause ‘that (who) is A’? Let us try the following paraphrases.

every\,\text{thing is B provided it is A}

\text{everything is such that it is B provided it is A}

Now we are getting somewhere, since this sentence divides as follows.

\text{everything is such that}
\text{it is B provided it is A}

Adding the crucial pronoun indices (variables), we obtain the following.

\text{everything x is such that}
\text{x is B provided x is A}

Recall ‘B provided A’ is equivalent to ‘B if A’, which is equivalent to ‘if A, then B’, which is symbolized \( A \rightarrow B \). Thus, the above sentence is symbolized as follows:

\[ \forall x(Ax \rightarrow Bx). \]

Note carefully the parentheses around the conditional; it’s OK to omit them when the formula stands by itself, but when it goes into making a larger formula, the outer parentheses must be restored. The same thing happens when we negate a conditional.

Of course, the corresponding formula without parentheses,

\[ \forall xAx \rightarrow Bx, \]

is also a formula of predicate logic, just as \( \sim A \rightarrow B \) is a formula of sentential logic. Both are conditionals. The latter says ‘if not A, then B’, in contrast to ‘it is not true that if A then B’, which is the reading of \( \sim(A \rightarrow B) \). The most accurate translation of the predicate logic formula, which is logically equivalent to

\[ \forall xAx \rightarrow By, \]

reads as follows.

if everything is A, then \textit{this} is B,

where ‘this’ points at something external to the sentence. This is a perfectly good piece of English, but it is definitely not the same as saying that every A is B.

Next, let us consider (s2) above. To say

some A is B,
for example, to say

some astronaut is brave,

is to say

there is at least one A that (who) is also B,

which is equivalent to

there is at least one A and it (he/she) is also B.

Notice that the pronoun ‘it’ points internally at ‘at least one A’.

We know how to say

draw is at least one A.
draw is at least one thing such that it is A

$\exists x A x$

How do we add the clause ‘that is also B’ or ‘and it is also B’? Well, we are saying that the thing in question is A, and we are saying in addition that it is B, so we are saying that it is A and it is B, which gives us the following.

draw is at least one thing such that
draw is A

and it is B

This is symbolized as follows.

$\exists x (A x \& B x)$

Notice once again that the outer parentheses are restored before the quantifier is prefixed. If we were to drop the parentheses, we obtain

$\exists x A x \& B x$,

which is logically equivalent to

$\exists x A x \& B y$,

which may be read

something is A, and this is B,

where ‘this’ points externally at whatever the person using this sentence is pointing toward. Although this is a perfectly good formula of predicate logic, it says something entirely different from ‘some A is B’

Next, let us consider (s3) above. To say

no A is B,

for example,

no astronaut is brave,
is to *deny* that there is at least one A who is B. In other words, it is the negation of ‘some A is B’, and is accordingly symbolized as follows,

\[ \sim \exists x (Ax \& Bx), \]

which is literally read as

it is not true that

there is at least one thing such that

it is A and it is B

Recall that \( \sim \exists x \mathcal{F} \) is equivalent to \( \forall x \sim \mathcal{F} \), for any formula \( \mathcal{F} \). In the above case, \( \mathcal{F} \) is the formula \( (Ax \& Bx) \), we have the following equivalence.

\[ \sim \exists x (Ax \& Bx) :: \forall x \sim (Ax \& Bx) \]

But, in sentential logic, we have the following equivalence (check the truth table!)

\[ \sim (A \& B) :: A \rightarrow \sim B \]

So, putting these together, we obtain the following equivalence.

\[ \sim \exists x (Ax \& Bx) :: \forall x (Ax \rightarrow \sim Bx) \]

Thus, we have an alternative way of formulating ‘no A is B’:

\[ \forall x (Ax \rightarrow \sim Bx), \]

which is read literally as

everything is such that

if it is A

then it is not B

Finally, let us consider (s4) above. To say

some A is not B

is to say

there is at least one A and it is not B,

which is symbolized very much the same way as ‘some A is B’,

\[ \exists x (Ax \& \sim Bx), \]

which is read literally as follows.

there is at least one thing such that

it is A and it is not B

Let us compare this with the following negation,

not every A is B,

which is symbolized just like
it is not true that every A is B,

thus:

\[ \sim \forall x (Ax \rightarrow Bx), \]

whose literal reading is

it is not true that

every thing is such that

if it is A then it is B.

Recall that \( \sim \forall x \phi \) is equivalent to \( \exists x \sim \phi \), for any formula \( \phi \); in the above case \( \phi \) is the formula \( (Ax \rightarrow Bx) \) – notice the parentheses – so we obtain the following equivalence.

\[ \sim \forall x (Ax \rightarrow Bx) :: \exists x \sim (Ax \rightarrow Bx) \]

But recall the following equivalence of sentential logic.

\[ \sim (A \rightarrow B) :: A \& \sim B \]

Thus, we have the following equivalence of predicate logic.

\[ \sim \forall x (Ax \rightarrow Bx) :: \exists x (Ax \& \sim Bx) \]

In other words, to say

not every A is B

is the same as to say

some A is not B.

For example, the following in effect say the same thing.

not every astronaut is brave
some astronaut is not brave