Example 9

(1) $\forall x \exists y Rxy$  
    $\text{Pr}$

(2) $\forall x \forall y [Rxy \rightarrow Rxx]$  
    $\text{Pr}$

(3) $\forall x [Rxx \rightarrow \forall y Ryx]$  
    $\text{Pr}$

(4) $\text{SHOW: } \forall x \forall y Rxy$  
    $\text{UD}$

(5) $\text{SHOW: } \forall y Ray$  
    $\text{UD}$

(6) $\text{SHOW: } Rab$  
    $\text{DD}$

(7) $\exists y Rby$  
    $1, \forall O$

(8) $\text{Rbc}$  
    $7, \forall O$

(9) $\forall y [Rby \rightarrow Rbb]$  
    $2, \forall O$

(10) $\text{Rbc} \rightarrow Rbb$  
    $9, \forall O$

(11) $\text{Rbb}$  
    $8, 9, \rightarrow O$

(12) $\text{Rbb} \rightarrow \forall y Ryb$  
    $3, \forall O$

(13) $\forall y Ryb$  
    $11, 12, \rightarrow O$

(14) $\exists x Fx$  
    $13, \forall O$

10. HOW EXISTENTIAL-OUT DIFFERS FROM THE OTHER RULES

As stated in the previous section, although we annotate existential-out just like other elimination rules (like $\rightarrow O$, $\lor O$, $\forall O$, etc.), it is not a true inference rule, but is rather an assumption rule. In the present section, we show exactly how $\exists O$ is different from the other rules in predicate and sentential logic.

First consider a simple application of the rule $\forall O$.

\[
\forall x Fx \\
\hline
Fa
\]

This is a valid argument of predicate logic, and the corresponding derivation is trivial.

(1) $\forall x Fx$  
    $\text{Pr}$

(2) $\text{SHOW: } Fa$  
    $\text{DD}$

(3) $\text{Fa}$  
    $2, \forall O$

Next, consider a simple application of the rule $\exists I$.

\[
\text{Fa} \\
\hline
\exists x Fx
\]

Again, the argument is valid, and the derivation is trivial.
The same can be said for every inference rule of predicate logic and sentential logic. Specifically, every inference rule corresponds to a valid argument. In each case we derive the conclusion simply by appealing to the rule in question.

But what about $\exists O$? Does it correspond to a valid argument? Earlier, I mentioned that, although the notation makes it look like $\forall O$, it is not really an inference rule, but is rather an assumption rule, much like the assumption rules associated with CD and ID.

Why is it not a true inference rule? The answer is that it does not correspond to a valid argument in predicate logic! The argument form is the following.

\[
\begin{align*}
\exists x Fx \\
\therefore F \alpha
\end{align*}
\]

In English, this reads as follows.

something is F
therefore, a is F

That this argument form is invalid is seen by observing the following counterexample.

(1) someone is a pacifist
(2) therefore, Adolf Hitler is a pacifist

If one has $\exists x Fx$, one is entitled to assume $F \alpha$ so long as ‘a’ is new. So, we can assume (for the sake of argument) that Hitler is a pacifist, but we surely cannot deduce the false conclusion that Hitler is/was a pacifist from the true premise that at least one person is a pacifist.

The argument is invalid, but one might still wonder whether we can nonetheless construct a derivation "proving" it is in fact valid. If we could do that, then our derivation system would be inconsistent and useless, so let’s hope we cannot!

Well, can we derive $F \alpha$ from $\exists x Fx$? If we follow the pattern used above, first we write down the problem, then we solve it simply by applying the appropriate rule of inference. Following this pattern, the derivation goes as follows.

\[
\begin{align*}
(1) & \exists x Fx & Pr \\
(2) & \text{SHOW: } F \alpha & DD \\
(3) & F \alpha & 1, \exists O \quad \text{WRONG!!!}
\end{align*}
\]

This derivation is erroneous, because in line (3) ‘a’ is not a permitted substitution according to the $\exists O$ rule, because the letter used is not new, since ‘a’ already
appears in line (2)! We are permitted to write down $F_b$, $F_c$, $F_d$, or a host of other formulas, but none of these makes one bit of progress toward showing $F_a$. That is good, because $F_a$ does not follow from the premise!

Thus, in spite of the notation, $\exists O$ is quite different from the other rules. When we apply $\exists O$ to an existential formula (say, $\exists x Fx$) to obtain a formula (say, $F_c$), we are not inferring or deducing $F_c$ from $\exists x Fx$. After all, this is not a valid inference. Rather, we are writing down an assumption. Some assumptions are permitted and some are not; this is an example of a permitted assumption (provided, of course, the name is new) just like assuming the antecedent in conditional derivation.

11. NEGATION QUANTIFIER ELIMINATION RULES

Earlier in the chapter, I promised six rules, and now we have four of them. The remaining two are tilde-existential-out and tilde-universal-out. As their names are intended to suggest, the former is a rule for eliminating any formula that is a negation of an existential formula, and the latter is a rule for eliminating any formula that is a negation of a universal formulas. These rules are officially given as follows.

\[
\text{Tilde-Existential-Out ($\sim \exists O$)}
\]
\[
\text{If a line of the form } \sim \exists v F[v] \text{ is available,}
\]
\[
\text{then one can infer the formula } \forall v \sim F[v].
\]

\[
\text{Tilde-Universal-Out ($\sim \forall O$)}
\]
\[
\text{If a line of the form } \sim \forall v F[v] \text{ is available,}
\]
\[
\text{then one can infer the formula } \exists v \sim F[v].
\]

Schematically, these rules may be presented as follows.

\[
\begin{array}{c}
\sim \exists O: \\
\sim \exists v F[v] \\
\hline
\forall v \sim F[v]
\end{array}
\]

\[
\begin{array}{c}
\sim \forall O: \\
\sim \forall v F[v] \\
\hline
\exists v \sim F[v]
\end{array}
\]
Before continuing, we observe is that both of these rules are *derived rules*, which is to say that they can be derived from the previous rules. In other words, these rules are completely *dispensable*: any conclusion that can be derived using either rule can be derived without using it. They are added for the sake of convenience.

First, let us consider \( \sim \exists O \), and let us consider its simplest instance (where \( F[v] \) is \( Fx \)). Then \( \sim \exists O \) amounts to the following argument.

**Argument 1**

\[
\begin{align*}
\sim \exists xFx & \quad \text{it is not true that there is at least one thing such that it is } F; \\
\therefore \forall x \sim Fx & \quad \text{therefore, everything is such that it is not } F.
\end{align*}
\]

Recall from the previous chapters that the colloquial translation of the premise is ‘nothing is F’, and the colloquial translation of the conclusion is ‘everything is unF’.

The following derivation demonstrates that Argument 1 is valid, by deducing the conclusion from the premise.

\[
\begin{align*}
(1) & \quad \sim \exists xFx & \text{Pr} \\
(2) & \quad \forall x \sim Fx & \text{UD} \\
(3) & \quad \sim Fa & \text{ID} \\
(4) & \quad Fa & \text{As} \\
(5) & \quad \times & \text{DD} \\
(6) & \quad \exists xFx & 1,6, \times \text{I} \\
(7) & \quad \times & 1,6, \times \text{I}
\end{align*}
\]

Next, let us consider \( \sim \forall O \), and let us consider the simplest instance.

**Argument 2**

\[
\begin{align*}
\sim \forall xFx & \quad \text{it is not true that everything is such that it is } F; \\
\therefore \exists x \sim Fx & \quad \text{therefore, there is at least one thing such that it is not } F.
\end{align*}
\]

Recall from the previous chapter that the colloquial translation of the premise is ‘not everything is F’ and the colloquial translation of the conclusion is ‘something is not F’.

The following derivation demonstrates that Argument 2 is valid. It employs (lines 1, 5, 11) a seldom-used sentential logic strategy.
In each derivation, we have only shown the simplest instance of the rule, where F[v] is Fx. However, the complicated instances are shown in precisely the same manner. We can in principle show for any formula F[v] and variable v that \( \neg \forall v F[v] \) follows from \( \neg \exists v F[v] \), and that \( \exists v \neg F[v] \) follows from \( \neg \forall v F[v] \).

Note that the converse arguments are also valid, as demonstrated by the following derivations.

Note carefully, however, that neither of the converse arguments corresponds to any rule in our system. In particular,

**THERE IS NO RULE TILDE-EXISTENTIAL-IN.**

**THERE IS NO RULE TILDE-UNIVERSAL-IN.**

The corresponding arguments are valid, and accordingly can be demonstrated in our system. However, they are not inference rules. As usual, not every valid argument form corresponds to an inference rule. This is simply a choice we make.
we only have negation-connective elimination rules, and no negation-connective introduction rules.

Before proceeding, let us look at several applications of \( \neg \exists O \) and \( \neg \forall O \) to specific formulas, in order to get an idea of what the syntactic possibilities are.

(1) \( \neg \exists x Fx \)

\[
\forall x \neg Fx
\]

(2) \( \neg \exists x (Fx \land Gx) \)

\[
\forall x \neg (Fx \land Gx)
\]

(3) \( \neg \exists x (Fx \land \forall y (Gy \rightarrow Rxy)) \)

\[
\forall x \neg (Fx \land \forall y (Gy \rightarrow Rxy))
\]

(4) \( \neg \forall x Fx \)

\[
\exists x \neg Fx
\]

(5) \( \neg \forall x (Fx \rightarrow Gx) \)

\[
\exists x \neg (Fx \rightarrow Gx)
\]

(6) \( \neg \forall x (Fx \rightarrow \exists y (Gy \& Rxy)) \)

\[
\exists x \neg (Fx \rightarrow \exists y (Gy \& Rxy))
\]

Having seen several examples of proper applications of \( \neg \exists O \) or \( \neg \forall O \), it is probably a good idea to see examples of improper applications.

(7) \( \neg (\exists x Fx \lor \exists y Gy) \)

\[
(\forall x \neg Fx \lor \exists y Gy)
\]

WRONG!!!

(8) \( \neg \exists x Fx \rightarrow \forall x Gx \)

\[
\forall x \neg Fx \rightarrow \forall x Gx
\]

WRONG!!!

In each example, the error is that the premise does not have the correct form. In (7), the premise is a negation of a disjunction, not a negation of an existential. The appropriate rule is \( \neg \forall O \), not \( \neg \exists O \). In (8), the premise is a conditional, so the appropriate rule is \( \rightarrow O \).

Of course, sometimes an improper application of a rule produces a valid conclusion, and sometimes it does not. (8) is a valid argument, but so are a lot of arguments. The question here is not whether the argument is valid, but whether it is an application of a rule. Some valid arguments correspond to rules, and hence do not have to be explicitly shown; other valid arguments do not correspond to
particular rules, and hence must be shown to be valid by constructing a derivation. Recall, as usual:

**INFERECE RULES APPLY EXCLUSIVELY TO WHOLE LINES, NOT TO PIECES OF LINES.**

(8) is valid, so we can derive its conclusion from its premise. The following is one such derivation. It also illustrates a further point about our new rules.

**Example 1**

\[\begin{align*}
(1) & \quad \neg \exists x Fx \rightarrow \forall x Gx & \text{Pr} \\
(2) & \quad \text{SHOW: } \forall x \neg Fx \rightarrow \forall x Gx & \text{CD} \\
(3) & \quad \forall x \neg Fx & \text{As} \\
\hline
(4) & \quad \text{SHOW: } \forall x Gx & \text{ID} \\
(5) & \quad \neg \forall x Gx & \text{As} \\
(6) & \quad \text{SHOW: } \neg & \text{DD} \\
(7) & \quad \neg \neg \exists x Fx & 1,5,\rightarrow O \\
(8) & \quad \exists x Fx & 7,\text{DN} \\
(9) & \quad Fx & 8,\exists O \\
(10) & \quad \neg Fx & 3,\forall O \\
(11) & \quad \neg & 9,10,\neg O
\end{align*}\]

This derivation is curious in the following way: line (4) is shown by indirect derivation, rather than universal derivation. But this is permissible, since ID is suitable for any kind of formula.

Indeed, once we have the rule \( \neg \forall O \), we can show any universal formula by ID. By way of illustration, consider Example 2 from Section 7, first done using UD, then done using ID.

**Example 2 (done using UD)**

\[\begin{align*}
(1) & \quad \forall x(Fx \rightarrow Gx) & \text{Pr} \\
(2) & \quad \text{SHOW: } \forall x Fx \rightarrow \forall x Gx & \text{CD} \\
(3) & \quad \forall x Fx & \text{As} \\
\hline
(4) & \quad \text{SHOW: } \forall x Gx & \text{DD} \\
(5) & \quad Fx & 1,\forall O \\
(6) & \quad Fx \rightarrow Ga & 3,\forall O \\
(7) & \quad Ga & 6,7,\rightarrow O
\end{align*}\]
Example 2 (done using ID)

(1) \( \forall x (Fx \rightarrow Gx) \)  
(2) \( \text{SHOW: } \forall x Fx \rightarrow \forall x Gx \)  
(3) \( \forall x Fx \)  
(4) \( \text{SHOW: } \forall x Gx \)  
(5) \( \neg \forall x Gx \)  
(6) \( \text{SHOW: } \bot \)  
(7) \( \exists x \neg Gx \)  
(8) \( \neg Ga \)  
(9) \( Fa \rightarrow Ga \)  
(10) \( Fa \)  
(11) \( Ga \)  
(12) \( \bot \)  

Now that we have \( \neg \forall O \), it is always possible to show a universal by indirect derivation. However, the resulting derivation is usually longer than the derivation using universal derivation. On rare occasions, the indirect derivation is easier; for example go back and try to do Example 1 using universal derivation.

We conclude this section with a derivation that uses \( \neg \forall O \) in a straightforward way; it also involves relational quantification.

Example 3

(1) \( \forall x (\forall y Rxy \rightarrow \neg \forall y Ryx) \)  
(2) \( \exists x \forall y Rxy \)  
(3) \( \text{SHOW: } \exists x \exists y \neg Rxy \)  
(4) \( \forall y Ray \)  
(5) \( \forall y Ray \rightarrow \neg \forall y Rya \)  
(6) \( \neg \forall y Rya \)  
(7) \( \exists y \neg Rya \)  
(8) \( \neg Rba \)  
(9) \( \exists y \neg Rby \)  
(10) \( \exists x \exists y \neg Rxy \)