CALCULATION OF MOMENT OF INERTIA  
WHEEL WITH MASSIVE SPOKES AND HEXAGONAL SYMMETRY

This appendix will outline the calculation of the moment of inertia, I, for a wheel where the spokes have mass. In addition, hexagonal symmetry will be assumed, so we can do the calculation for 1/6 of the wheel. A diagram of 1/6 of the wheel is shown in the figure.

The wheel is solid for $0 \leq r \leq f_1R$ (where $R$ is the outer diameter of the wheel and $f_1$ is a fraction between 0 and 1). The spokes are in the space $f_1R \leq r \leq f_2R$. The wheel is solid again in the region $f_2R \leq r \leq R$.

At a radius $r = f_1R$ the spoke covers an angular range $0 \leq \theta \leq \theta_1$, and at radius $r=f_2R$ it covers an angular range $0 \leq \theta \leq \theta_2$. For convenience, we set $\theta_1 = a$ and $\theta_2 = b$. We also assume that the range of $\theta$ goes linearly from $a$ to $b$ as $r$ goes from $f_1R$ to $f_2R$. This means that at some $r$ in the space $f_1R \leq r \leq f_2R$ the range of $\theta$ is given by $\theta_{range} = \alpha + \frac{r-f_1R}{f_2R-f_1R} (b-a)$

The angular range is calculated from

$$\sin \left( \frac{\theta}{2} \right) = \frac{L/2}{R} \Rightarrow \theta = 2\sin^{-1} \left( \frac{L}{2R} \right)$$

The scheme is to first calculate $I$, then calculate $M$, and finally divide $I$ by $MR^2$ so we can express $I$ as a fraction of $MR^2$. We assume a uniform density, $\rho$.

$$I = \int r^2 dm.$$  But, $dm = \rho r dr d\theta$.  So, $I = \rho \int r^3 dr \int d\theta$

We will evaluate $I$ in sections:  
- calculate $I_1$ for $0 \leq r \leq f_1R$
- calculate $I_2$ for $f_1R \leq r \leq f_2R$
- calculate $I_3$ for $f_2R \leq r \leq R$

$$I_1 = \rho \int_{0}^{f_1R} r^3 dr \int_{\theta_0}^{\theta_1} d\theta = \rho \cdot \frac{\pi}{3} \cdot \frac{(f_1R)^4}{4}$$

because we are only looking at 1/6 of the wheel.

$$I_3 = \rho \int_{f_2R}^{R} r^3 dr \int_{\theta_0}^{\theta_1} d\theta = \rho \cdot \frac{\pi}{3} \cdot \frac{R^4 - (f_2R)^4}{4}$$

the range of $\theta$ is $0 \rightarrow \frac{2\pi}{6} = \frac{\pi}{3}$
Then total \( I = I_1 + I_2 + I_3 \) is given by

\[
I = \rho \cdot R^4 \cdot \frac{\pi}{12} \cdot f_1^4 + \rho \cdot R^4 \cdot \frac{\pi}{12} \cdot (1 - f_1^4) + \rho \cdot R^4 \cdot \frac{\pi}{4} \left( f_2^4 - f_1^4 \right) + \rho \cdot R^4 \cdot \frac{b - a}{f_2 - f_1} \left( f_2^5 - f_1^5 \right) - f_1 R \left( f_2^4 - f_1^4 \right)
\]

We now calculate the quantity \( MR^2 \), and as with \( I \), evaluate this in sections. Calculate:

\[
M_1 = MR^2 \text{ for } 0 \leq r \leq f_1 R
\]

\[
M_2 = MR^2 \text{ for } f_1 R \leq r \leq f_2 R
\]

\[
M_3 = MR^2 \text{ for } f_2 R \leq r \leq R
\]
Then total MR\(^2\) = M_1 + M_2 + M_3 is given by

\[
\text{MR}^2 = \rho \cdot R^4 \cdot \frac{\pi}{6} \cdot f_2^2 + \rho \cdot R^4 \cdot \frac{\pi}{6} \cdot (1 - f_2^2) + \rho \cdot R^4 \cdot \frac{a}{2} \cdot (f_2^2 - f_1^2) + \rho \cdot R^4 \cdot \left(\frac{b - a}{f_2 - f_1}\right) \left[f_2^3 \left(\frac{f_2}{3} - f_1^2\right) - f_1^3 \left(\frac{1}{3} - f_1^2\right)\right]
\]

Now take the ratio \(l/\text{MR}^2\), it is independent of \(\rho\) and \(R\):

\[
l/\text{MR}^2 = \frac{\frac{\pi}{12} \left[f_2^4 - f_2^2 + 1\right] + \frac{\alpha}{4} \left[f_2^4 - f_1^4\right] + \left(\frac{b - a}{f_2 - f_1}\right) \left[f_2^3 \left(\frac{f_2}{5} - f_1^2\right) + f_1^3 \left(\frac{1}{4} - \frac{1}{5}\right)\right]}{\frac{\pi}{6} \left[f_2^2 - f_1^2 + 1\right] + \frac{\alpha}{2} \left[f_2^2 - f_1^2\right] + \left(\frac{b - a}{f_2 - f_1}\right) \left[f_2 \left(\frac{f_2}{3} - f_1\right) + f_1 \left(\frac{1}{2} - \frac{1}{3}\right)\right]}
\]

Two trivial checks:

a) For a solid wheel, we expect \(l/\text{MR}^2 = \frac{1}{2}\). For solid wheel set \(a = b = \pi/3\). Then \(b - a = 0\) and we have:

\[
l/\text{MR}^2 = \frac{\frac{\pi}{12} \left[f_2^4 - f_2^2 + 1\right]}{\frac{\pi}{6} \left[f_2^2 - f_1^2 + 1\right]} = \frac{\pi}{12} \cdot \frac{\pi}{6} = \frac{1}{2}
\]

b) For a rim only, we expect \(l/\text{MR}^2 = 1\). In this case we have \(a = b = 0\), \(f_1 = 0\) and \(f_2 = 1\) and

\[
l/\text{MR}^2 = \frac{\frac{\pi}{12} \left[1 - f_2^2\right]}{\frac{\pi}{6} \left[1 - f_2^2\right]} = \frac{6}{12} \left[1 - f_2^2\right] \left(1 + f_2^2\right) = \frac{6}{12} \left(1 + f_2^2\right) = \frac{1}{2} \left(1 + 1\right) = 1 \quad \text{as} \quad f_2 = 1
\]